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A brief review on my studies: managing the complexity on using Linking Visual Active Representations (LVAR)

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Abstract

Investigating, managing, reflecting on, reading, reorganizing, conceiving, activating and implementing new ideas, notions and terms in the context of an extended research study lie at the core of creativity and innovation. The most important step after you have coined a new idea is to investigate the idea's potential to be transformed into a successful implementation in class and in education more generally. In this paper, I present a brief review on the studies I have conducted since 2005 into the notion of linking visual active representations. In this context, I will also present a table which lists the new terms I have coined in association with the DGS environments, and listing the publications in which they were reported. In conclusion, I present my views on the contribution Dynamic Geometry systems have made to the construction of knowledge and to the learning and teaching of mathematics.

Keywords: Linking Visual Active Representations, Dynamic Geometry environments (DGS), managing, conceiving and activating knowledge

1. Introduction: Why and when do we have to define new terms?

In the following excerpt, Scheiner (2017) presents the ways in which manipulating and reflecting on objects can create and/or transform ideas or extract and give meaning to extant ideas: "*Extracting meaning and giving meaning can occur simultaneously: an individual might extract meaning by manipulating objects and reflecting on the actual instances of such objects, while at the same time an individual gives meaning to the instances that appear in the senses by activating and attaching ideas. With respect to giving meaning, an individual might either activate already available ideas to attach meaning to instances or an individual might create new ideas in the moment by transforming ideas to gain new insight that allows attaching new meaning to an object of consideration*" (Scheiner, 2017, p.149).

I have continued to investigate mathematical objects using Dynamic Geometry systems (DGS) down the years, and remain deeply devoted to this kind of research. It has helped me to 'see' notions for which I had to coin new expressions, "small pieces of knowledge", since I could not find them described using existed phrases. In other words, my research into DGS actuated me to conceive, define and develop various new terms and notions--all of which have been incorporated, by means of Dynamic Geometry environments, into the field of the Didactics of Mathematics.

During the years of producing these pieces of knowledge, by writing and investigating mathematical meanings, I was simultaneously reading several theories, trying to deeply understand what has been introduced by other researchers [as well as, to re-organize my existing knowledge]. Biehler, Scholz, Sträßer, and Winkelmann (1994), in the Preface of their substantial study "*Didactics of Mathematics as a Scientific Discipline",* argue that "*the boundary between scientific work and (constructive) practice is – to say the least – 'fuzzy'*" (p.3). In my opinion (Patsiomitou, 2019c, p.5): *Didactics of Mathematics* is the science and art of teaching and learning mathematics, designing and implementing teaching and instructional products for the learning of mathematics in static or computing environments- in synchronous or asynchronous teaching- incorporating the content of the subject of mathematics, mathematics pedagogy, the history of mathematics, and psychological theories of learning, teaching and human-computer interactions. The *Didactics of Mathematics using DGS* is a complex scientific area which focuses mainly at the interaction between two main systems: *the system of teaching, and the system of learning mathematical objects, using DGS*. The *Didactics of Geometry using DGS*, as part of the Didactics of Mathematics, focuses on the didactic methods of geometry, using multiple representational systems, incorporating dynamic tools in synchronous or asynchronous teaching.

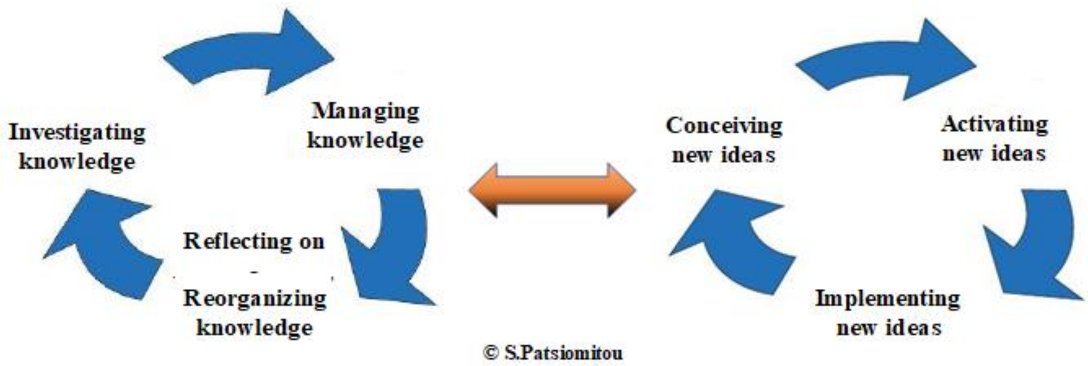


Fig. 1. The cycle of inspiration for new ideas and new terms

In the *Critique of Pure Reason*, Kant (1929/1965) argues that "knowledge is constituted of both sensual perceptions and concepts" (see also, Radford, 2002). This phrase recognizes and underlines the force of his argument, that: There can be no doubt that all our knowledge begins with experience. […] *knowledge consists in the determinate relation of given representations to an object*" (see also, Radford, 2005). In Fig. 1, I present the cycle of the inspiration process: investigating and managing knowledge leads to its being continually reorganized and reflected upon, leading to the formation of a point of view, the conceiving of new ideas, their activation and hence their implementation. Then the process can repeat itself in a constantly interacting dipole. A cycle of interaction between preexisting and new knowledge.

Do we have to define new notions? In my opinion, as research goes about its scientific work, new notions and theories will inevitably be developed.

2. Managing, conceiving, activating and introducing "dynamic" notions

The table below incorporates the new terms which I coined, listing the publications in which they were reported and used for the first time. The earlier chronology includes the introduction of the notion. The notions I adapted are also presented in the same table.

A statement of the terms is included below as they are contained in my published articles, or as they have been subsequently adapted as the research has evolved.

Table 1: Dynamic notions

	Dynamic Notion	Citations
1	Linking "alive" Representations	Patsiomitou, 2005a, 2005b, 2006a, b, c, d, e, f, g, 2008f, g
		Patsiomitou, 2007a, 2018a, b, 2019c
2	A "dynamic" method of exhaustion for number pi (π)	Patsiomitou, 2006f, 2007c
		Patsiomitou, 2018a, 2019c
3	The "dynamic" building of number fi (φ) as an approximation process	Patsiomitou, 2006g, 2007e
		Patsiomitou, 2019b, c
4	Structural Algebraic Units	Patsiomitou, 2006 (oral presentation in Greek), 2007e, 2008e, 2009b ; Patsiomitou, 2008c, 2009a, 2019c
5	Linking Visual Active Representations (LVAR)	Patsiomitou, 2008a, b, 2010; Patsiomitou, 2008h; Patsiomitou & Koleza, 2008, 2009
6	Reflective Visual Reaction	Patsiomitou, 2008a, b; Patsiomitou & Koleza, 2008, 2009
7	Design and definition of LVAR modes (five phases)	Patsiomitou, 2008b, 2010, 2019c ; Patsiomitou & Emvalotis 2009b
		Patsiomitou, 2012a, d, 2020a, 2022a
8	A proposal for a DG research -based curriculum	Patsiomitou, 2010 ; Patsiomitou & Emvalotis 2009d, 2010a, b
		Patsiomitou, 2012a, 2020a, 2022a
9	Custom tools (scripts) as alive/active objects	Patsiomitou, 2005a, 2006c, d, e, g, 2007b, d, e, 2009b, c, 2022a
		Patsiomitou, 2008d, 2018a, 2019c; Patsiomitou & Emvalotis 2009c
10	Theoretical and Experimental dragging	Patsiomitou, 2011a, 2012b, 2014, 2018b, 2019c
		Patsiomitou, 2011b, 2012α, 2015a, c, 2016a, b, 2020a, 2021c, 2022a

11	Dynamic point, Dynamic segment, Parametrical segment, Dynamic propositions, dynamic meaning	Patsiomitou, 2011a, 2012b, 2014, 2018b, 2019c
		Patsiomitou, 2011b, 2012α, 2015a, c, 2016a, b, 2020a, c, 2021a, b, c, 2022a
12	Instrumental decoding, instrumental obstacle	Patsiomitou, 2011a, 2012b, 2014, 2018b, 2019c
		Patsiomitou, 2011b, 2012α, 2013b, 2015a, c, 2016a, b, 2020a, b, c, 2021c, 2022a
13	Discursive, visual and operational apprehension of tool's use (adaptation of Duval's (1995) cognitive notions)	Patsiomitou, 2011a, 2012b, 2014, 2018b, 2019c
		Patsiomitou, 2011b, 2012α, 2015a, c, 2016a, b, 2020a
14	"House of parallelograms" (adaptation of Graumann's (2005) "*house of quadrilaterals*")	Patsiomitou, 2012b, 2019c
		Patsiomitou, 2012α, 2015α, c, 2020a
15	A classification of the internally constructed quadrilaterals	Patsiomitou, 2012b, 2019c
		Patsiomitou, 2012α, 2015α, c, 2020a
16	A proposal for a qualitative upgrading of math curricula (including fractals, spirals etc.)	Patsiomitou, 2005a, b, 2007a, b, c, d, 2009b, c, d, c, e, f, g, h, 2012a, 2015b, d, 2016c
		Patsiomitou 2007a, 2012b, c
17	A Dynamic Hypothetical Learning Path (Adaptation to Simon's (1995) *Hypothetical Learning Trajectory*)	Patsiomitou, 2012b, 2019c;
		Patsiomitou, 2012α, 2015α, c, 2020a, b, c, 2022a
18	Introduction of a Pseudo-Toulmin model	Patsiomitou, 2011a, 2012b, 2014, 2018b, 2019c, 2021a, b
		Patsiomitou, 2011b, 2012α, 2015a, c, 2016a, b, 2020a, 2022a
19	Dynamic (/ perceptual) definition and arbitrary	Patsiomitou, 2013a

	economic definition. (An adaptation of the Govender & De Villiers' (2004) clarification)	Patsiomitou, 2012α, 2015a, c, 2016a, b, 2020a
20	An adaptation of Battista (2007)'s categorization regarding the development of students' abstract processes	Patsiomitou, 2013a, 2019c
		Patsiomitou, 2012α, 2015a, c, 2016a, b, 2020a
21	Linking Visual Active Representations (LVAR)-reformulation of the definition	Patsiomitou, 2012b, 2014, 2019c
		Patsiomitou, 2012a, 2020a
22	Dynamic Didactic cycle (Adaptation to Simon's (1995) *The Mathematics Teaching Cycle*)	Patsiomitou, 2014, 2019c, 2021a, b
		Patsiomitou, 2015α, β, 2022a
23	Implementation of LVARs for the teaching of mathematics in class	Patsiomitou, 2014, 2019c
		Patsiomitou, 2012c, d, 2015d, 2016a, b, 2020a, b, c, d
24	Kinds of Transformations	Patsiomitou, 2014, 2019c, 2021a, b
		Patsiomitou, 2009a, b, 2021c, 2020a, b, c, 2022a
25	The meaning of "alive" tool	Patsiomitou, 2005a, b, 2006b, 2020a, 2022a
		Patsiomitou, 2018a, 2019c, 2022c
26	Dynamic Active Learning Trajectory	Patsiomitou, 2018a, 2019c
27	Dynamic object, Dynamic Diagram, Dynamic section,	Patsiomitou, 2019 a, b, c Patsiomitou, 2020a, 2022a
28	Hybrid-dynamic objects	Patsiomitou, 2006a, 2022a
		Patsiomitou, 2019 a, b, c, 2021a, b
29	Procept-in-action	Patsiomitou, 2019 b, c
30	Empirical Classification Model for Sequential Instructional Problems in Geometry	Patsiomitou, 2019a, c
		Patsiomitou, 2022a
31	Representation and dynamic representations	Patsiomitou, 2019c
		Patsiomitou, 2020a, 2022a
32	Didactics of Mathematics	Patsiomitou, 2019c
		Patsiomitou, 2020a, b, d, 2022a

33	Instrumental Learning Trajectories	Patsiomitou 2021a, b
	interdependence/intra-dependence between dynamic tools and objects	Patsiomitou, 2021c, 2022a
34	Virtual Cuisenaire Rods	Patsiomitou, 2022 b, c, 2023
35	Virtual Froebel Gifts	Patsiomitou, 2022d, 2023
36	Virtual Cuisenaire Rods (online Web sketchpad version)	in collaboration with Prof. Scher, 2022

Investigating, managing, reflecting on, reading, reorganizing, conceiving, activating and implementing new ideas, notions and terms in the context of an extended research study lie at the core of creativity and innovation. The most important step after you have coined a new idea is to investigate the idea's potential to be transformed into a successful implementation in class and in education more generally.

3. Theoretical interpretation of the defined and introduced terminology

In the current section, I shall list the terminology as it is defined and introduced, also providing a brief theoretical interpretation of the terms.

Linking 'alive' representations are representations that enable students *to focus their attention on simultaneous modifications (and transformations) of objects on the screen* (Patsiomitou, 2005a, p. 68). They can be linked /connected through transformations of dynamic diagrams.

Custom tools are 'alive' encapsulated objects (Patsiomitou, 2018b, p.51) created in a DGS environment that operate as a referent point for organizing, retrieving and reversing information (Patsiomitou, 2008d), and thus facilitating the anticipation and manipulation of the *instrumented action schemes* during an *instrumental genesis* process. A custom tool can become a medium for students' cognitive development and a tool to develop their abstract thought. The customized tools of the Geometer's Sketchpad (Jackiw, 1991) allow users to create 'alive' tools.

An example of linking 'alive' representations: In my experiments, when I was designing dynamic problems or tasks in the DGS environment (e.g., Patsiomitou, 2005a, in Greek) I noticed that when a dynamic representation was selected as a new tool, it lit up and remained lit until it was moved to a new location on the desktop. Let's look at the next screenshot (Fig. 2a, b). Activating the new tool from its initial building and repeating it at a different point on the Desktop creates an alive object that remains alive until it is repositioned.

A first and very important effect on students' thinking stems from the Sketchpad software allowing the user to create *sequential linking pages* so that the whole Sketchpad file becomes an "*alive book*" (Patsiomitou, 2005a, p. 63, in Greek; Patsiomitou, 2014). I conceived the LVAR notion when creating linking or linked pages in Sketchpad files to model fractals for my research (Patsiomitou, 2005a).

I implemented dynamic–active, or 'alive', linking representations in the design of problems, activities and tasks during my teaching in secondary schools, using dynamic means (e.g., Patsiomitou, 2005a, 2006a, b, c, d, e, f, g, 2007b, c, d, e). I shall mention two important factors I was taking into account when designing the activities, both of which are based on classroom observations made over many years of teaching mathematical meanings. Namely, I used the software's interaction techniques to link the steps in the constructional, transformational or explorative actions or processes, linking simultaneously the steps in the proof via a sequence of dynamic linking pages or the steps in the same page in the DGS environment.

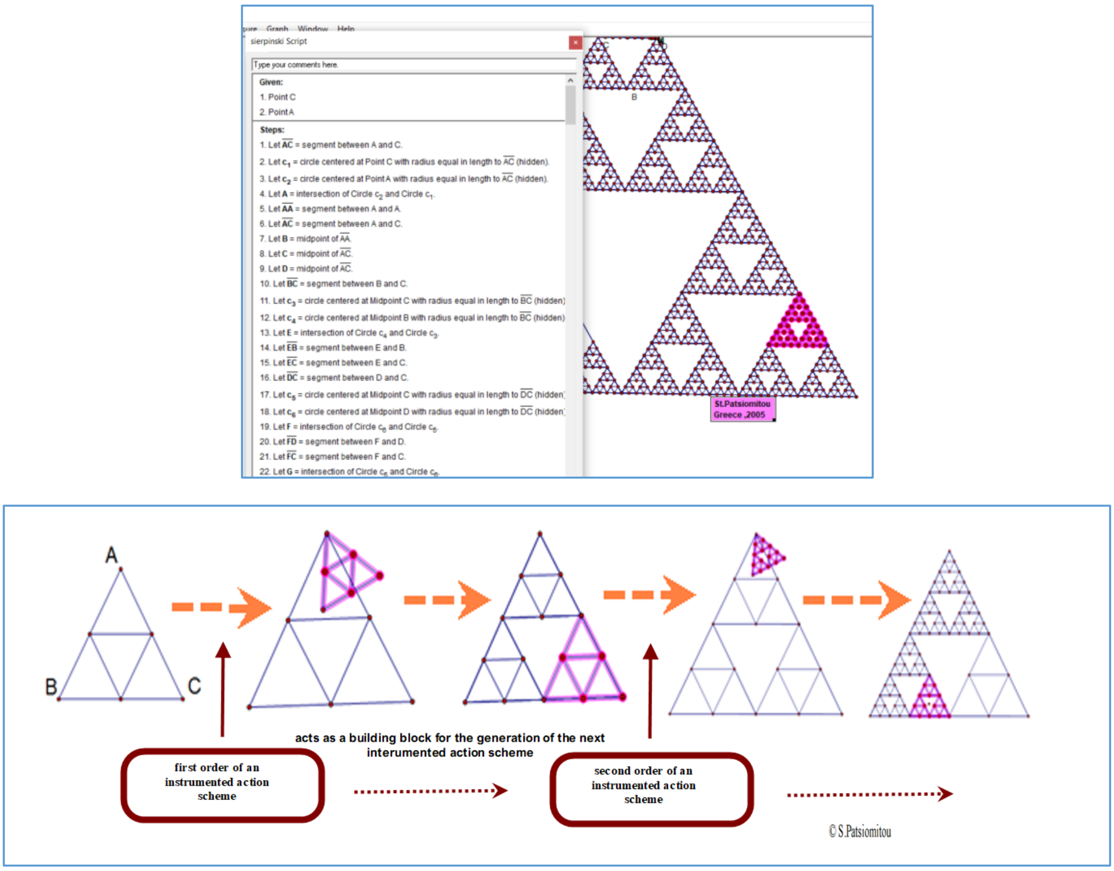
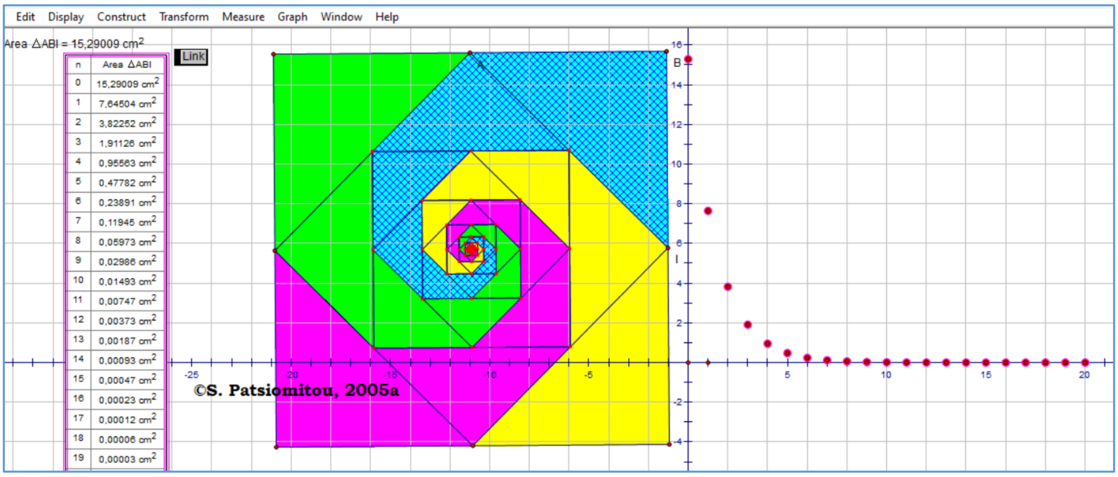


Fig. 2a, b. Sierpinski's triangle building (Patsiomitou, 2005a, in Greek) via the utilization of successive custom tools with increasing constructive complexity (see also, Patsiomitou, 2014, p. 26)

Both factors impact directly on how students are guided to the proving process through solving problems, and hence on how students are guided to abstract thought processes. Thus, during the problem-solving process or when reproducing a theorem with a view to proving it, the teacher or students ask questions which help them construct the proof. Thus, a problem would be solved by breaking it down into a series of questions. This process is a qualitative evolution of the Socratic method ("maieftiki" in Greek) by which teachers ask questions designed to elicit the correct answer and reasoning processes. What I had verified was my student's reinvention of pieces of knowledge, presented in dynamic geometry software (DGS), created with this **new kind of representations.** These active representations made different students dynamically reinvent and **discover concepts in several different ways, at different times over the years**. *The students consequently had an environment of* ***linking active representations*** *in which the shape of the fractal had been linked with the table of the measurements via the functional* *process of iteration, which continuously was linked with the graphic representation of the sequence via the plotted points*. (Patsiomitou, 2005a, see also, Patsiomitou, 2007a, 2014).

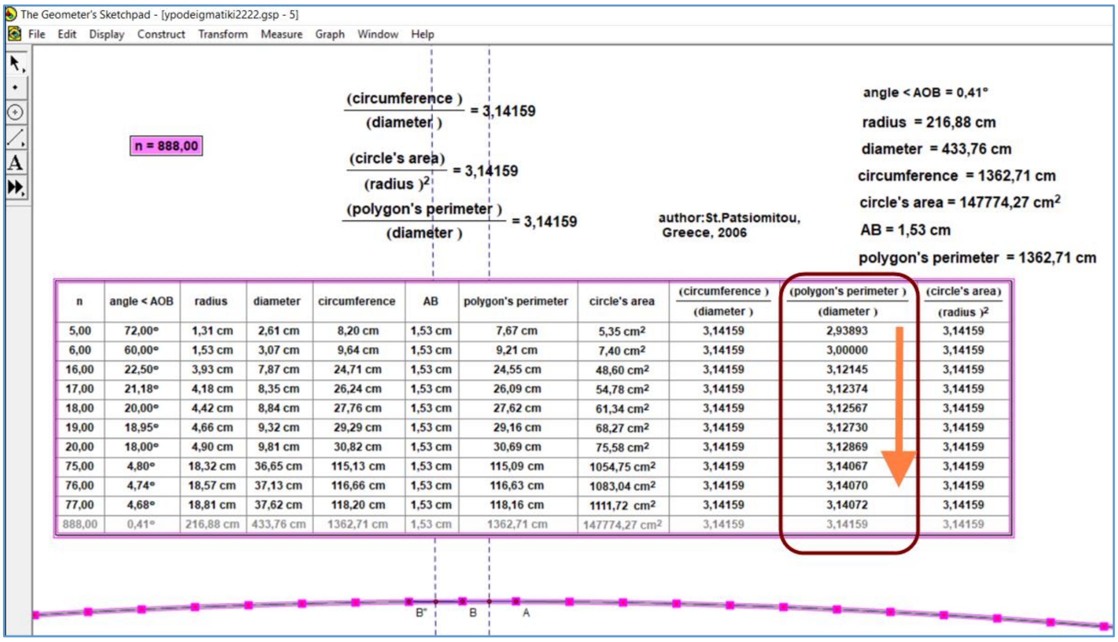


n	Area ΔABI
0	15,29009 cm²
1	7,64504 cm²
2	3,82252 cm²
3	1,91126 cm²
4	0,95563 cm²
5	0,47782 cm²
6	0,23891 cm²
7	0,11945 cm²
8	0,05973 cm²
9	0,02986 cm²
10	0,01493 cm²
11	0,00747 cm²
12	0,00373 cm²
13	0,00187 cm²
14	0,00093 cm²
15	0,00047 cm²
16	0,00023 cm²
17	0,00012 cm²
18	0,00006 cm²
19	0,00003 cm²

Fig. 3. Baravelle Spiral's LVAR for young learners (e.g., Patsiomitou, 2005a,2009b)

A "dynamic" method of exhaustion for number pi (*π*)

Exhaustion is a method invented by Eudoxus (407-355 B.C.E.) (Euclid's Elements, Book XII). Archimedes (287–212 B.C.E.) used the method of exhaustion, inscribing inside a circle a sequence of regular polygons whose areas converge to the area of the circle. Archimedes realized that the true value of pi would be obtained by "exhausting".



n	angle < AOB	radius	diameter	circumference	AB	polygon's perimeter	circle's area	(circumference) / (diameter)	(polygon's perimeter) / (diameter)	(circle's area) / (radius)²
5,00	72,00°	1,31 cm	2,61 cm	8,20 cm	1,53 cm	7,67 cm	5,35 cm²	3,14159	2,93893	3,14159
6,00	60,00°	1,53 cm	3,07 cm	9,64 cm	1,53 cm	9,21 cm	7,40 cm²	3,14159	3,00000	3,14159
16,00	22,50°	3,93 cm	7,87 cm	24,71 cm	1,53 cm	24,55 cm	48,60 cm²	3,14159	3,12145	3,14159
17,00	21,18°	4,18 cm	8,35 cm	26,24 cm	1,53 cm	26,09 cm	54,78 cm²	3,14159	3,12374	3,14159
18,00	20,00°	4,42 cm	8,84 cm	27,76 cm	1,53 cm	27,62 cm	61,34 cm²	3,14159	3,12567	3,14159
19,00	18,95°	4,66 cm	9,32 cm	29,29 cm	1,53 cm	29,16 cm	68,27 cm²	3,14159	3,12730	3,14159
20,00	18,00°	4,90 cm	9,81 cm	30,82 cm	1,53 cm	30,69 cm	75,58 cm²	3,14159	3,12869	3,14159
75,00	4,80°	18,32 cm	36,65 cm	115,13 cm	1,53 cm	115,09 cm	1054,75 cm²	3,14159	3,14067	3,14159
76,00	4,74°	18,57 cm	37,13 cm	116,66 cm	1,53 cm	116,63 cm	1083,04 cm²	3,14159	3,14070	3,14159
77,00	4,68°	18,81 cm	37,62 cm	118,20 cm	1,53 cm	118,16 cm	1111,72 cm²	3,14159	3,14072	3,14159
888,00	0,41°	216,88 cm	433,76 cm	1362,71 cm	1,53 cm	1362,71 cm	147774,27 cm²	3,14159	3,14159	3,14159

Fig. 4. A "dynamic" exhaustion for pi (π) using active representations (Patsiomitou, 2006f, 2018a)

Following Archimedes, I introduced a "*dynamic"* method of *exhaustion*, an approaching process to obtain number pi. This method is described extensively in my papers (Patsiomitou, 2006f, 2007c; Patsiomitou, 2018a, 2019c). As I have written (Patsiomitou, 2018a, p. 232): "In Fig.4 we can view a n-gon with 888 sides along with the tabularized measurements and calculations. The length of the side remains unmodified, whereas the iteration process changes the number of sides. As we can see in the final column of the table in Fig. 3, if the number of sides is 5 the ratio of the n-gon's perimeter to its diameter is almost equal to 2, 93. If we continue increasing the number of sides (for example, to 96 or 888 sides) we will see that the approximation process results in a number that is almost equal to number pi (≅ 3,14159)".

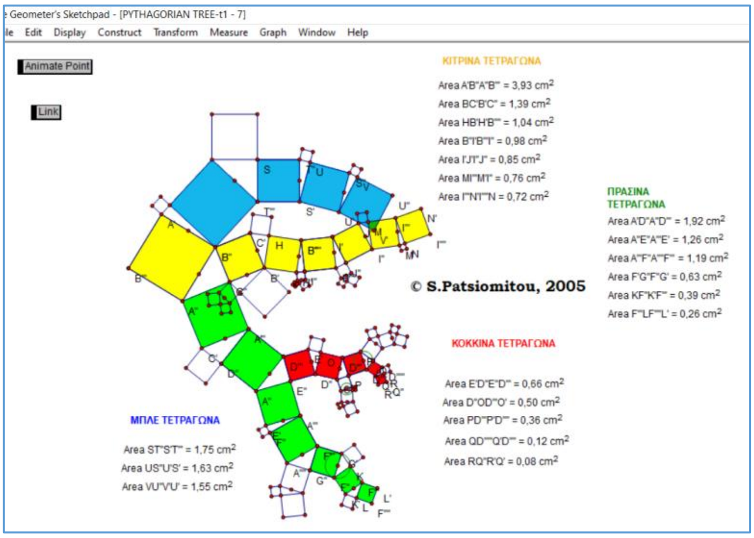


Fig. 5. Games with custom-tools **"**Pythagorean trees" for young learners

I was led to the theoretical constructs of the notions "Linking Visual Active Representations" and "Reflective Visual Reaction" as a result of my experience designing DGS tasks and problems for my Master's dissertation (January-May 2005)- and Ph.D. thesis (January-May 2007) and the research I conducted using these tools, as reported in the following excerpt from the study "*The development of students' geometrical thinking through Linking Visual Active Representations*"(Patsiomitou & Koleza, 2009): "For this reason, the researcher (S.P.) designed the multiple pages of the software using interaction techniques such as 'hide/show action buttons' or 'link-buttons'(p. 159)[…] "the meanings of Linking Visual Active Representations, and Reflective Visual Reaction during a dynamic geometry problem solving, are introduced / defined in the present study directly connected with the design process in the software as follows" (p. 162).

Linking Visual Active Representations (Patsiomitou, 2008a, p.365) are the successive phases of the dynamic representations of the problem which link together the problem's constructional, transformed representational steps in order to reveal an ever increasing constructive complexity; since the representations build on what has come before, each one is more complex, and more integrated than in previous stages, due to the student's (or teacher's, in a half-preconstructed activity) choice of interaction techniques during the problem-solving process, aiming to externalize the transformational steps they have visualized mentally (or exist in their mind).

- **Reflective Visual Reaction** (Patsiomitou, 2008a, p.365) is that reaction which is based on a reflective mode of thought, derived from interaction with LVAR in the software, thus complementing and adding to the student's pre-exesting knowledge or facilitating comprehension and integration of new mathematical meanings.

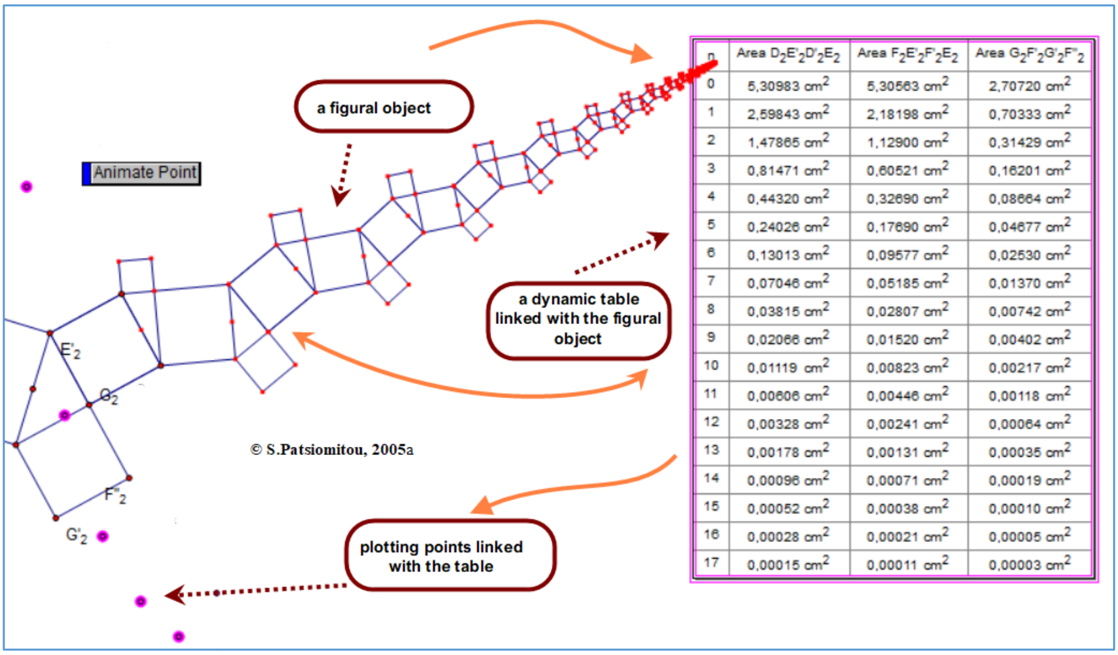
Even though these definitions were incomplete, they were still accepted for publication at three conferences during roughly the same period, having first been published (June, 2008) in the following extensive article (Patsiomitou, 2008a): "*The development of students' geometrical thinking through transformational processes and interaction techniques in a dynamic geometry environment*". Linking Visual Active Representations (LVARs) are Visual Mathematical Representations (VMRs) which are *dynamic*, *linking* and *active;* LVARs can help students in the proving process. As it is aforementioned, I designed the LVAR modes in a successive sequence of increasing complexity. The terms that I chose to define the concept of LVARs have been illustrated in previous studies (Patsiomitou, 2008a, b; Patsiomitou, 2010, p.2):

- The term '*linking*' was preferred to 'linked' because the former denotes something that can be linked, but is not necessarily linked at this moment.

- All DGS objects are necessarily '*visual*' representations of what they stand for.

- An '*active*' representation is a representation that causes action, motion or change because it is in operation, in effect or in progress. Dynamic representations can always *become active* if we cause an action on them. LVAR *always involve semi pre-constructed* dynamic diagrams that can be linked and become active in accordance with the wishes of the user, meaning the user is not limited to "actions pre-set by the sketch creator".

The term "*active representation"* is considered in mindful processing of information in which students individually or in collaboration manipulate and interact with the objects and tools in the dynamic environment and construct their knowledge by reflecting on what they have created.



n	Area $D_2E'_2D'_2E_2$	Area $F_2E'_2F'_2E_2$	Area $G_2F'_2G'_2F''_2$
0	5,30983 cm^2	5,30563 cm^2	2,70720 cm^2
1	2,59843 cm^2	2,18198 cm^2	0,70333 cm^2
2	1,47865 cm^2	1,12900 cm^2	0,31429 cm^2
3	0,81471 cm^2	0,60521 cm^2	0,16201 cm^2
4	0,44320 cm^2	0,32690 cm^2	0,08664 cm^2
5	0,24026 cm^2	0,17690 cm^2	0,04677 cm^2
6	0,13013 cm^2	0,09577 cm^2	0,02530 cm^2
7	0,07046 cm^2	0,05185 cm^2	0,01370 cm^2
8	0,03815 cm^2	0,02807 cm^2	0,00742 cm^2
9	0,02066 cm^2	0,01520 cm^2	0,00402 cm^2
10	0,01119 cm^2	0,00823 cm^2	0,00217 cm^2
11	0,00606 cm^2	0,00446 cm^2	0,00118 cm^2
12	0,00328 cm^2	0,00241 cm^2	0,00064 cm^2
13	0,00178 cm^2	0,00131 cm^2	0,00035 cm^2
14	0,00096 cm^2	0,00071 cm^2	0,00019 cm^2
15	0,00052 cm^2	0,00038 cm^2	0,00010 cm^2
16	0,00028 cm^2	0,00021 cm^2	0,00005 cm^2
17	0,00015 cm^2	0,00011 cm^2	0,00003 cm^2

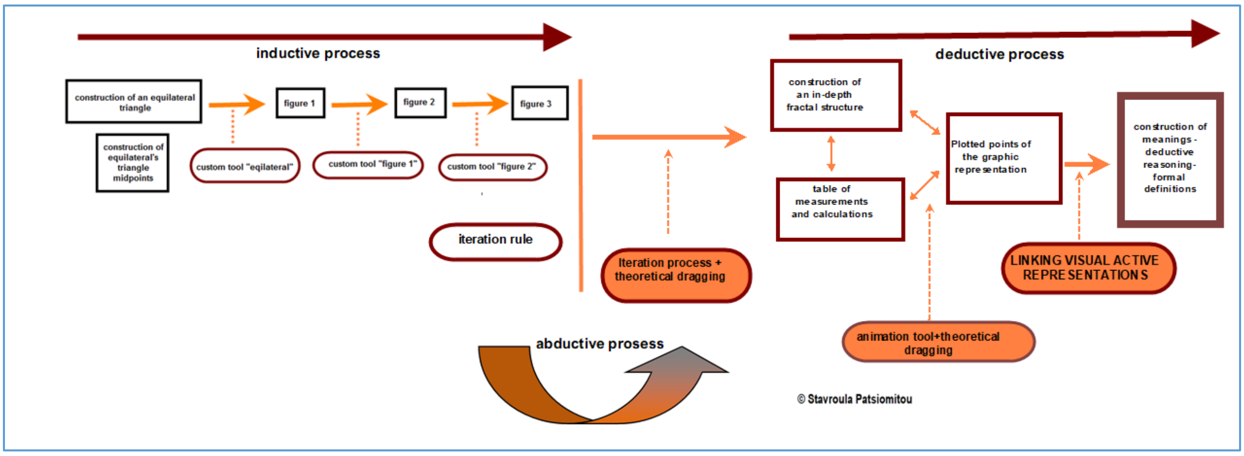


Fig. 6a, b. A Pseudo Toulmin model explaining the birth of the meaning 'Linking Visual Active Representations' (See also, Patsiomitou, 2014, p. 29)

A part of the research with active representations incorporated Euclid's identities, which I designed in the DG environment.

Structural Algebraic Units are linking visual active algebra-tiles that can also serve as algebraic units (as can, for instance, the terms x^2, x, 1), allowing the form of an identity —or, more generally, a polynomial—to be constructed. They are also used to represent integers (or constants) and variables.

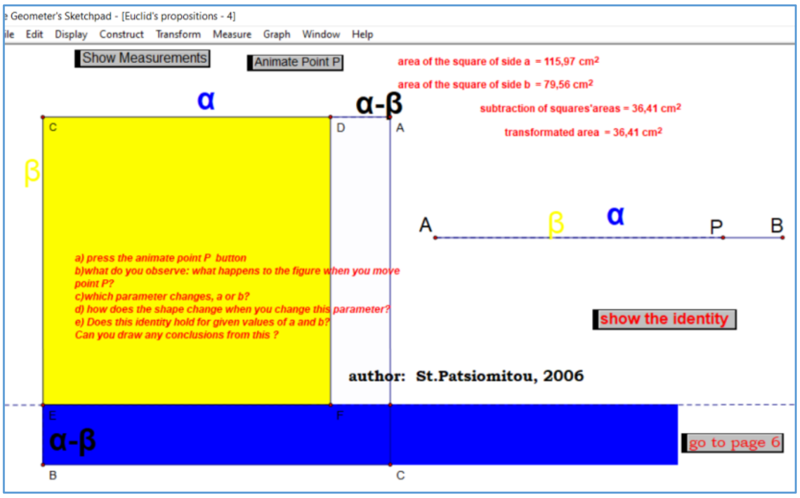


Fig. 7. The "dynamic" Fifth Proposition of Euclid's Elements (e.g., Patsiomitou, 2007e) with covariations of the active tools

Figure 7 is a snapshot of a diagram that I created in Geometer's Sketchpad to represent in an interactive way the Proposition 5, in Euclid's Elements. The conceptualization of its construction is reported in details in my paper "Do geometrical constructions affect students' algebraic expressions?" (Patsiomitou, 2008c) and its improved version "The Impact of Structural Algebraic Units on Students' Algebraic Thinking in a DGS Environment" (Patsiomitou, 2009a).

- **Definition of LVAR's Modes** (Patsiomitou, 2008b, 2010).

In my study "*Building LVAR (Linking Visual Active Representations) modes in a DGS environment"* (Patsiomitou, 2008b, 2010) the process of proving a problem or theorem consists of a series of steps which can function as responses, anticipating the questions posed explicitly or implicitly by teacher or student. These steps can be expressed with the proper design of LVAR modes, using different software tools for every mode.

Mode A-the inquiry/information mode: In this mode of LVARs the students familiarize themselves with the field under investigation using the instantiated parts of the diagrams which lead them to discover a certain structure. As I have written (Patsiomitou, 2019a, p.18): "*The synthesis of the dynamic representation incorporates an image that is a permanently annotated pictorial representation, a two-dimensional hybrid object representing the closed and curved polygonal island, annotated in green and two dynamic fractal trees placed on the island. The background (font) of the screen has been selected to be light-blue using the complex preferences pop-up menu, to give the impression of the sea around the island. The positions of the trees P, O are two points with zero degrees of freedom*".

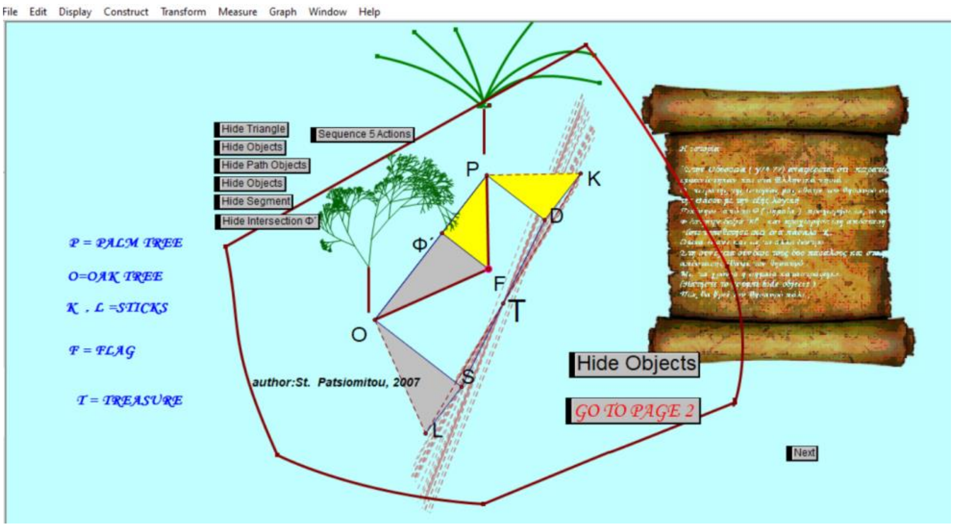


Fig. 8. Lost treasure problem -Mode A (Patsiomitou, 2008b, p. 169)

Mode B-the directed orientation mode: The sequential constructional and transformational steps of LVARs are displayed as a global shape to which more elements and/or information are gradually added, rearranging, annotating and probing parts of the diagram when action buttons are directly manipulated. The steps in the construction of the diagrammatic reconstruction which are displayed by pressing the action buttons are linked to suitable questions and their answers. The process has the following advantages: during this process the students are led to cognitively connect additional, complementary, transformational reconstructions of the problem configuration and actions aimed at externalizing the student's thoughts by means of suitable chain questions which guide them towards the solution to the problem.

Mode C–the explicitation mode: Transformations in increasingly complex linked dynamic and active representations of the same phase of the problem modify the on-screen configurations simultaneously. The figures on screen undergo a *metamorphosis* as a result of the manipulations. The translation gives to the dynamic representation the property to a simultaneous alteration of every dynamic object on them if we drag any point.

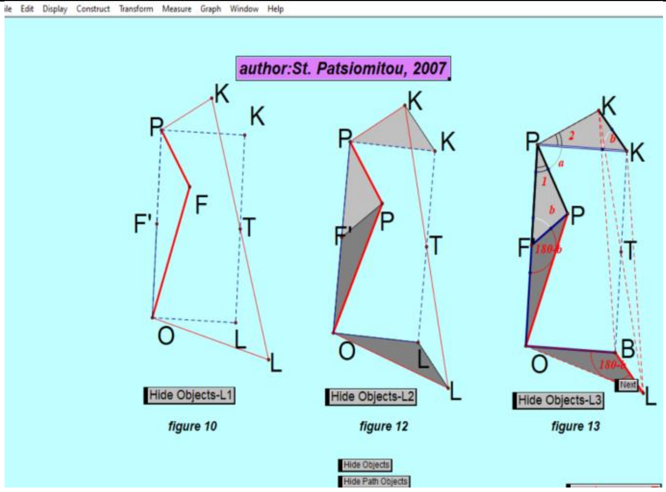


Fig. 9a. Mode C–the explicitation mode (Patsiomitou, 2008a, p. 373, 2008b, p. 172)

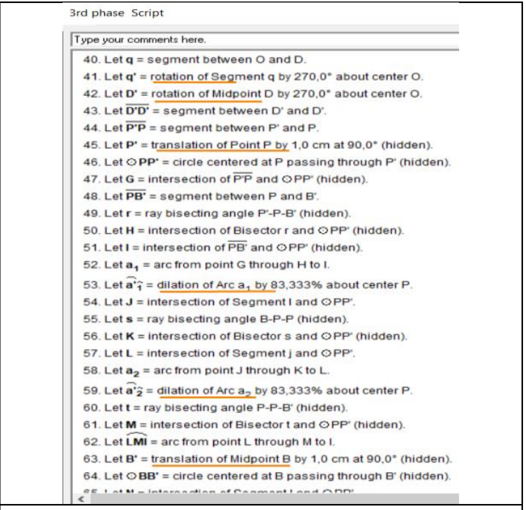


Fig. 9b. A custom tool for the Mode C

As I have written (Patsiomitou, 2019a, p.19): "*This is a complex phase. The dynamic diagrams are linked, using a translation transformation and every diagram on the right is a sequential successive and gradual procedure conducted on the previous one which is on the left. The translation gives to the dynamic representation the property to a simultaneous alteration of every dynamic object on them if we drag any point. The synthesis of the dynamic LVARepresentation has the following design: Point F has two degrees of freedom and point O has 0 degrees of freedom. The screen background has been changed using Sketchpad's complex preferences dialogue in order to link it to the previous page. The experimental dragging of point V does not transform the rectangle's figure, which remains a hybrid object on screen*".

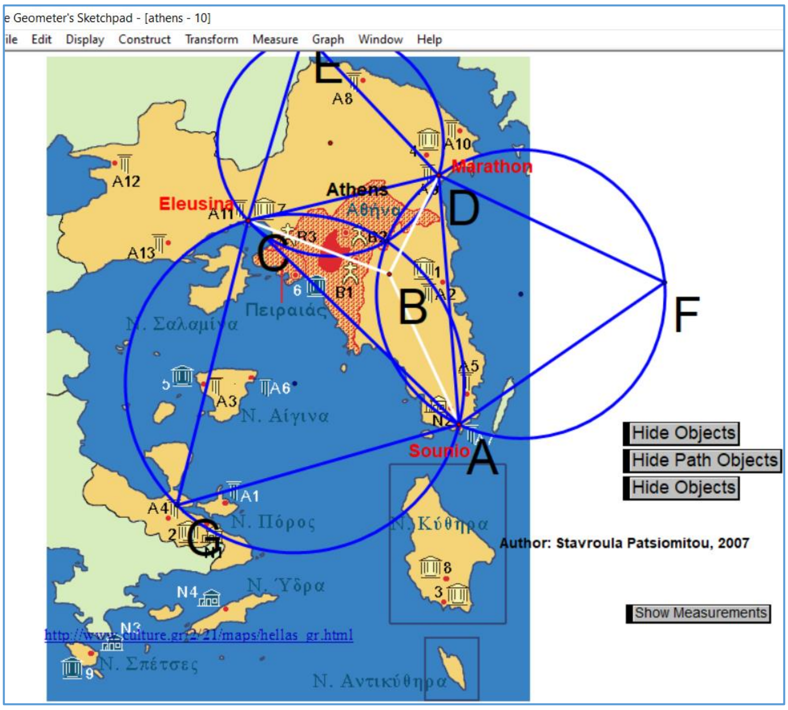


Fig. 10: The solution to the problem in the final modeled representation (Patsiomitou, 2008b, 2010)

Mode D–the free orientation mode: LVARs can be displayed side by side on the same page of the software in an overview. The students can focus their observation on what extra information is presented. in the next emerging iconic form of the representation. The emerging additional representations can be dragged independently. The students are led to a proof that confirms their initial reasoning, conjectures and exploratory processes.

Mode E–the integration mode: Successive LVARs on different pages that are linked cognitively, compose the solution to the problem. The information with which they became familiar in the new network of evoked geometrical objects and their interrelationships is reviewed and summarized. The students have developed thinking processes and applied skills, developing a mathematical model to interpret the realistic problem. **Semperasma** (Patsiomitou, 2008a, b): *the building and transforming of the semi-predesigned LVAR leads the students to pass from a visual way of thinking to a theoretical geometrical one, or to pupils' mental transformations. Students use verbal formulations to exchange their ideas meaning that they transform their mental objects into a language mapping, corresponding to LVAR transformations on pages in the software.*

- **A re-conceptualized, research-based DG curriculum using Geometer's Sketchpad**

For the purposes of my Ph.D. thesis (2007-2012), I investigated how a hypothetical learning path for quadrilaterals would (a) help my students develop their van Hiele levels, and (b) help me formulate a proposal for the reconceptualization of the geometry curriculum enriched with DG tasks and problems. A short description of this proposal was reported in the paper "*Composing and testing DG research–based curriculum designed to develop students' geometrical thinking*" (Patsiomitou& Emvalotis, 2009d, p.1) as follows: "One of the researchers (S.P) conceived and applied a re-conceptualized, research-based DG curriculum using Geometer's Sketchpad environment as part of her Ph.D. thesis, which is still in progress. She was responsible for the choice of activities, for session planning and student assessment. It was drawn up and implemented in the following way: phases connected with the structuring of the conceptual content and the order in which the interaction techniques of the software were introduced, both of which were made progressively more difficult during the course of the instruction. The DG curriculum featured four strands: 1) students build figures with an emphasis on the "construction" menu 2) students build figures through symmetry with an emphasis on the "transform" menu 3) the exploration of open ended problems, aiming to help students investigate problems employing and consolidating the tools and meanings acquired in the two previous faces; 4) building and transforming semi predesigned Linking Visual Active Representations (LVAR) -a combination of different transformational processes using software's interaction techniques- for the investigation of their impact on students' cognitive processes with regard to the conjecturing and proving processes depending on their level". I briefly reported the phases of the hypothetical learning path in my study "*Building LVAR (Linking Visual Active Representations) modes in a DGS environment*" (Patsiomitou, 2008b, 2010, p.1). The same description is incorporated in the paper "*The development of students' geometrical thinking through a DGS reinvention process*" (Patsiomitou & Emvalotis, 2010b, p.34, volume 4).

- **The Dynamic Hypothetical Learning Path** (DHLP)

A Dynamic Hypothetical Learning Path (DHLP) (Patsiomitou, 2012b) is a hypothetical learning path through the dynamic geometry software, *supported by LVARs*. The phases of the DHLP are reported in the detailed and long article "*A Linking Visual Active Representation DHLP for student's cognitive development*" (Patsiomitou, 2012b, p. 58) as follows:

"The phases of the DHLP are interconnected in terms of: a) the conceptual context, b) the order in which the software's technological tools are introduced, and c) the increasing difficulty at both levels. This description of the DHLP is a synthesis of an *instructional design process and a redesign process*. It is based on the research I conducted from January to May, 2007.

- In the *instructional design process*, I describe how I predicted the hypothetical transitional understanding of the meaning of parallelograms and the students' way of thinking during the solution of the problems in combination with their actions in the software with the closest possible approach.
- In the *instructional redesign process*, I describe the procedures that demanded the addition of new tools, which helped the students of the experimental group overcome cognitive and instrumental obstacles that they faced during the research process".

A hypothetical learning path is a cognitive tool based in social constructivism. Furthermore, the learning paths are dynamic, when instructional DG (Dynamic Geometry) activities are incorporated. A Dynamic Hypothetical Learning Path (DHLP) can incorporate real-world problems or simulations of problems in the DGS environment that had been analysed and designed in terms of (a) the students' van Hiele (vH) levels of thinking, starting from the lower vH levels to elicit higher vH levels, (b) their sequential conceptual content, and (c) the student's comprehension of the links between representations and mathematical meanings conceptually and procedurally. The DHLP mentioned above is consisted with the following phases and the problems posed and encountered during these phases (Patsiomitou, 2012b, 2019c, pp. 180-188)

Table 2: The phases of the DHLP

Phase A: Building and transforming quadrilaterals through Linking Visual Active Representations	The aim of the first phase of the research process was for the students to obtain the competence to build and transform linking structurally unmodified representations of parallelograms. Problem 1, 2, 3, 4: Construction of a parallelogram. Construction of a rectangle. Construction of a rhombus. Construction of a square.
Phase B: Investigating and building figures through symmetry	In this phase the notion of symmetry ant their properties are introduced by using the transformations of the rotation and reflection of the software. The recognition/understanding of the symmetry of geometrical objects is the fundamental aim of this study, in accordance with van Hiele's (1984, 1986) theory. I separated the second phase into four subphases: Part B1. The recognition—visualization part of the second phase Problem: Reflect point A (on a given line l) in order to construct its image, point A΄. Part B2. The perceptually componential analysis part of the second phase Problem: Construct the axes of symmetry of rectangle.

	Problem: Construct the axes of symmetry of rhombus. Then join the midpoints of the opposite sides with a segment and explain why it is an axis of symmetry or not. Then, drag the vertex of the rhombus to form a square. Problem: Construct a square' axes of symmetry. Part B3. The informal componential analysis part of the second phase Redesign process: The investigation of the meaning of rotational symmetry Redesign process: The example and the counter-example of custom tool's use. Redesign process: The construction of the *structure of the bisected diagonals* Part B4. The formal componential analysis part of the second phase.
Phase C: Investigation of theorems presented as dynamic problems (i.e., Varignon's theorem, Viviani's theorem) aiming the students to construct the classification of quadrilaterals with regard to their diagonals	The students of all groups investigated several instances of Varignon's theorem occurring from the use of dragging (Patsiomitou, 2012a). Also, they investigated several instances of Viviani's problem, in order to construct formal proofs and generalizations (still unpublished). In my study "*A Linking Visual Active Representation DHLP for student's cognitive development*" (Patsiomitou, 2012b), I describe **a new classification of quadrilaterals**, due to the different properties of the internal parallelograms (or middle-quadrilaterals) which are constructed if we join the midpoints of the external quadrilaterals
Phase D: The LVAR modes in correspondence to the learning phases are described	An extended study which describes the research conducted with LVAR modes is included in my paper: "*Linking Visual Active Representations and the van Hiele model of geometrical thinking*" (Patsiomitou, 2008b), as well as the revised version of this paper: "*Building LVAR (Linking Visual Active Representations) modes in a DGS environment*" (Patsiomitou, 2010).

The design of the LVAR Modes led me to later categorize the tools and define them using a distinct code, as follows (Patsiomitou, 2012a, 2013a, p. 803):

Table 3: The categorization of the tools during the phases of the DHLP	
T1 for the point used by experimental dragging	Phase A
T2 for the point used by theoretical dragging	
T3 for the reflection tool	Phase B
T8 for the circle tool	
T9 for the rotation tool	
T11 for the parametric tool	Phase C
T12 for the custom tool "symmetry"	
T15 for the custom tool used with 'economy or catachresis'	
T16 for the hide/show action button tool	Phase D

T17 for the trace tool	
T18 for the annotation tool	
LVAR synthesis of tools	

Furthermore, the characteristics of the van Hiele levels that appeared during analysis of students' dialogues led me to create an adaptation to Battista's (2007) categorization. Concretely, I conceived and applied the following process: The first column of an Excel matrix contained the tools and the synthesis of the tools that helped students to formulate an expression or a characteristic that could be an indication of their van Hiele level. Particularly, the first row of an Excel matrix contains characteristics of the van Hiele levels, each with a distinct code (Patsiomitou, 2012b). For example, characteristics of level 1, 2.1 […] 3.4 were coded as follows: I0 for cognitive conflicts and I1 for informal descriptions, etc. described at the Table 3 below.

Table 4: An adaptation to Battista's (2007) categorization on van Hiele levels (Patsiomitou, 2012a, 2013a, p.804)	
level 1	I0 for cognitive conflicts and I1 for incorrect and informal descriptions.
level 2.1	I2 for dynamic perceptual definition and I3 for the synthesis of formal and informal descriptions of students.
level 2.2	I4 for incomplete definitions and incomplete reasoning and I5 for inductive argumentation/concepts-in-action or theorems-in-action.
level 2.3	I6 for formal description and non-economical definitions and I7 for connections between meanings.
level 3.1	I8 for economical definitions and I9 for logical correlations between meanings.
level 3.2	I10 for structural analysis competence, I11 for abductive-deductive reasoning.
level 3.3	I12 for deductive argumentation, I13 for the generic example proof scheme.
level 3.4	I14 for thought experiment proof scheme, I15 for the competence of logical hierarchy.

- **Theoretical and Experimental dragging**

I described these kinds of dragging as follows (Patsiomitou, 2011a,b): "I consider there to be two main diacrises in dragging utilizations with regard to students actions: (a) *the theoretical dragging* in which the student aims to transform a drawing into a figure on screen, meaning s/he intentionally transforms a drawing to acquire additional properties and (b) the *experimental dragging* in which the student investigates whether the figure (or drawing) has certain properties or whether the modification of the drawing in the picture plane through dragging leads to the construction of another figure (or drawing)".

- **Instrumental decoding**

I support the following from the empirical results of my investigations (e.g., Patsiomitou, 2011a, p. 362, 2011b): The construction of a dynamic diagram in a DGS environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation ("construct a parallelogram" for example) into a mental image, which is to say an internal representation recalling an archetype / prototype image (e.g., Hershkovitz, 1990, Presmeg, 1992) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. This process requires the student to decode their actions using software primitives, functions etc. In order to accomplish a construction in the software the student must acquire the competence for *instrumental decoding* (Patsiomitou, 2011, p. 362) meaning the competence to transform his/her mental images to actions in the software.

- **Discursive, visual and operational apprehension of tool's use** (an adaptation to Duval's (1995) semiotic analysis)

Competence in the DGS environment depends on the competence of the cognitive analysis which students bring to bear when decoding the utilization of software tools, based on Duval's (1995) semiotic analysis of students' apprehension of a geometric figure. During the development of a construction, I think that the student has to develop three kinds of apprehension when selecting software objects which accord with the types of cognitive apprehension outlined by Duval (1995, pp.145-147) namely *perceptual, sequential, discursive, and operative apprehension*. In concrete terms, the *competence of instrumental decoding* in the software's constructions depends on: a) the *sequential apprehension* of the tools selection; b) the *verbal apprehension* of the tools selection which means the student has to verbalize this process, and c) *a place way type of elements operation* on the figure due to his/her perceptual apprehension. Then s/he has constructed the *operative apprehension* of the figure's elements for the construction, meaning the competence to operate the construction.

- **Introduction of a Pseudo-Toulmin model** (Patsiomitou, 2011a, 2012b, p. 57)

Argumentation of students can be represented using Toulmin's (1958) model. I have introduced and explained *pseudo-Toulmin's model* through examples in which (a) the data could be an element or an object of the dynamic diagram, and (b) a warrant could be a tool or a command that guarantees the result which is the claim (or the resulted formulation).

- **Dynamic point, Dynamic segment, Parametrical segment, Dynamic propositions, dynamic meanings**

A 'dynamic' point is a fundamental element in a dynamic construction. In figure 1, a screenshot of a point is illustrated along with the resizing of the same point in Snagit software (on the left). As we can see the point has brightness and thickness. It is a dynamic "active" point or an "alive" point (Patsiomitou, 2005a, 2006c, 2018b).

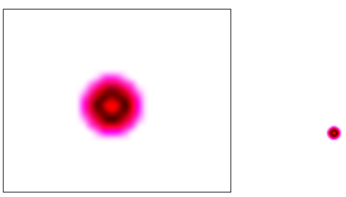


Fig. 11a. Screenshots of an active dynamic point in DGS

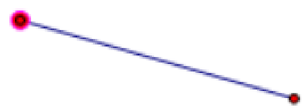


Fig. 11b. An active and a non-active dynamic point on the same segment

In the Sketchpad software a dynamic point is a tool. Consequently, a dynamic point can be defined in correlation with (a) its degrees of freedom and (b) its brightness, making it an alive, active tool. In my studies, I also defined the term dynamic segment as follows (Patsiomitou, 2011a, p.365, 2019c, p.84):

- **Dynamic' segment** is a segment made in a DGS. The 'dynamic' segment is a portion of a straight line which does not consist of points. Dynamic points can be placed independedly on the dynamic segment and move free with one degree of freedom on the path to which they belong. You can move a segment from its active point.
- **A parametric segment, a parametric figure.** It is a segment (or a figure) created with the use of parameters. The parametric figures are created with the use of parametric segments. These parametrical segments can be transformed dynamically by transforming (e.g., by using animation) the parameters with which they have been created, meaning the parental objects in a continuous/or not process (Patsiomitou, 2019b, p.40):
- Firstly, the animation on parameters turns the dynamic diagram to a more detailed and complex representation than the one we have created using the tools (e.g., segments, lines and circles).
- Secondly, the concept of parameters belongs to algebra. On the other hand, when we create a figure in a static environment, we never use a parameter to create the figure, just as we never define a segment as a parameter for use in our construction. Moreover, animating the parameters transforms the synthesis of the diagram into an "infinite" number of snapshots, which the user would probably not consider manipulating by her/himself.
- **Dynamic objects, diagrams and sections** (Patsiomitou, 2019a, p.15):

A dynamic geometrical object is every object that has been constructed in a dynamic geometry software interface. This object could be a "drawing" or a "figure" which intrinsically has dynamic properties. This definition is complementary to what Gonzalez, and Herbst (2009) argue regarding the dynamic diagram as "a diagram made with DGS and that has the potential to be changed in some way by dragging one or more of its parts" (p.154).

A dynamic diagram is an external representation composed out of a set of rationally related dynamic objects in a DGS environment. A dynamic diagram can be a simulation of a problem modelled in the DGS environment, which includes many geometric objects and combinations of interaction techniques implemented in these objects.

A dynamic section is a set of dynamic diagrams that are linked to each other procedurally and conceptually, even if they may differ structurally. A dynamic section contains meanings belonging to the same class that are united or joined into a whole, which in the concrete situation symbolically means they exist in one ["alive" book] section or they are dynamically linked.

- **Dynamic reinvention of knowledge** is the kind of knowledge the students could reinvent by interacting with the artefacts made in a DGS environment, knowledge for which they themselves are responsible (Patsiomitou, 2012b, p. 57).

Finally, "Linking Visual Active Representations" (LVARs) during a dynamic geometry problem solving session are defined as follows (e.g., Patsiomitou, 2012a, b, 2019a), incorporating the notion of *instrumental decoding* (Patsiomitou, 2011a, b, 2012 a, b) and the notion of *dynamic hypothetical learning path (DHLP)* (Patsiomitou, 2012a, b, 2019c, p. 196):

- **Linking Visual Active Representations** are the successive/consequential building steps in the dynamic representations of a problem or between problems, which repeat the same procedural steps or steps reversing a procedure in the same phase or between different phases of a *hypothetical learning trajectory*. LVARs reveal an increasing structural complexity by conceptually and structurally linking the transformational steps taken by the user (conducting anticipatory thought experiments) through the interaction techniques provided by the software as a result of his/her development of thinking and understanding of geometrical concepts, which are *instrumentally decoded* by the way s/he has visualized mentally what exist in his/her mind or a revision of it.
- **Reflective Visual Reaction** is the reaction based on a reflective mode of thought, derived from interaction with LVARs in the software.
- **Definition of the notion of Representation (**Patsiomitou, 2019c, p.42)

A representation is both (a) an external entity (such as a verbal expression, a graph, a figure, a map, a picture), which is to say an external correspondence of objects or processes with the objects that are represented by the entities brought into being as representing objects by the modelling process, and (b) an internal mental entity, meaning a structurally equivalent modification of physical/mental objects/processes which are constructed in the mind, as a result of the processing/elaboration of information and the manipulation of objects and concepts due to the cognitive schemes which have developed in the subject's mind".

- **Dynamic Active Leaning trajectories** are sequential instructional tasks and activities engaged in [with] a learning goal and designed [with dynamic active representations] to engender mental linking representations which help students develop their thinking in the specific math domain.
- **Procept-in-action**

Gray & Tall (1991) define the meaning of 'procept' as a combination of the words "pro-[cess] + [con]-cept", The meaning of an elementary procept is according to them "an amalgam of […]: a process which produces a mathematical object and a symbol which is used to represent either process or object […]" (Gray & Tall, 1994). Building on the above, I think there is a continuous process ongoing in students' mind as they create a concept.

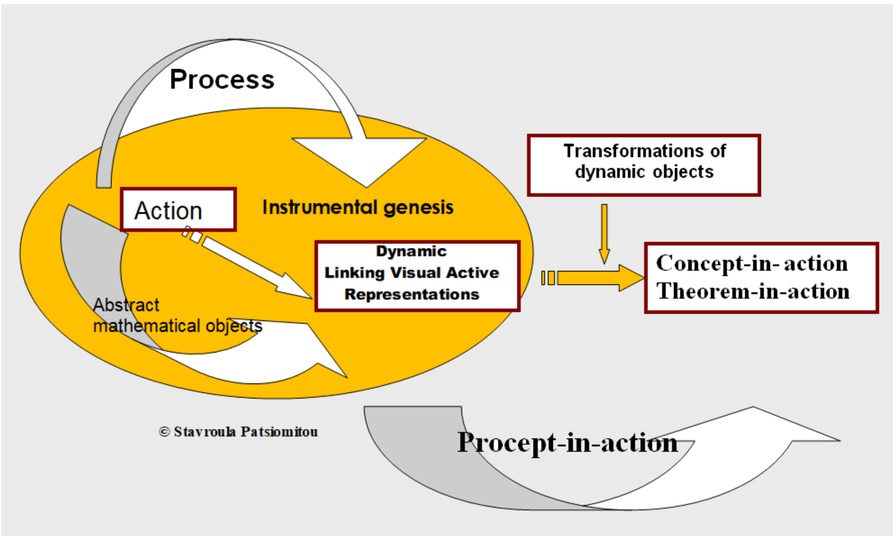


Fig. 12. A diagram of a procept-in-action (Patsiomitou, 2019b, p.)

The meaning of 'procept' is thus dynamic in a DGS environment; adapting its meaning to a '*procept-in-action*' for the DGS environment could thus support the appearance of *operational invariants* (concepts-in-action and theorems-in-action) during the problem-solving situation and the students' actions on a dynamic object or a dynamic representation/diagram.

The transformation of all the objects in a DGS environment, leads the students to conceive the *unaltered properties of the mixed entity***.** They can express a *concept-in-action or theorem-in-action, through* the reification of mathematical objects and the interiorization of the process of dynamic movement, counting and dragging the segments: this is a *procept-in-action*, meaning a process which leads to a concept-in-action or theorem-in-action.

D*ynamic perceptual definition* (Patsiomitou, 2012b) is the term for the process by which the student informally 'defines' a geometrical object by using the tools of the software.

- **A categorization of geometrical problems**

My work with students at the secondary and tertiary levels led me to identify *five types of geometrical problems* (Patsiomitou, 2019a, p.3). Summarizing, I would like to present five investigational levels of a problem-solving process, synthesizing, elaborating on and addressing conceptual and procedural understanding through feedback provided at every intermediate step in the problem's solution which is designed in the light of the cognitive processes elicited at each level. Thus, constitutes a *cognitive trajectory* through problem solving for the students' cognitive development (Patsiomitou, 2019a, p. 19):

- The *first level of sophistication* is that of open problems using materials (e.g., squared papers, dot papers, or several means, including DGS). This phase can be extensive by means of DGNA (*Dynamic geometrical problems with non-given answers*) problems using sequential dynamic LVAR representations. When a student is engaged with the activity of solving a problem modeled by dynamic LVARepresentations s/he connects that activity with both the product and the thought process during investigational process. LVARs scaffold students' mental processes such as perception, information recall and reasoning. Students can also discover the solution through active experimentation.
- The *second level* comes after the introduction of "big ideas" or "core ideas" (Battista, 2011). During this phase, the teacher can use DGGA (*Dynamic geometrical problems with given answers)* problems posed for investigation and proof in a DGS environment. The students can mentally combine structural properties of conceived cognitive processes.
- The *third level* is that of real world HGNA (*Dynamic geometrical problems modeled in a DGS with hybrid–dynamic geometrical representations*) problems which are modeled in a DGS environment using dynamic or hybrid-dynamic representations. A teacher can support students' reasoning by giving them other immediate problems which will scaffold the theoretical background required by the problem as they investigate all the possible or multiple solutions to the problem. They can also investigate a concrete situation of the hybrid-dynamic representations, choosing to give to the parameter's concrete magnitudes.
- The *fourth level* will be that of RGNA (*Real world geometrical problems with non-given answers)* problems, accepting a challenge and trying to reinvent the solution. The students at this level must have the conceptual and procedural competence to investigate the problem. At this level, the problem cannot be solved by some routine procedures.

• *The fifth level* will be that of the problem in a SGGA (*Static geometrical problems with given answers*) problem in a static environment. This is the level with the higher degree of difficulty. This is why students are not able to solve static geometry problems, when they belong at the lower van Hiele levels.

The emerging theoretical construct provides both a methodology for building up the problem-solving process and an approach to addressing difficulties students face in learning geometrical concepts, which uses anticipatory thought experiments in which we envision how we can construct an organizational structure and a learning trajectory through problem solving as the students engage with the process.

- **Instrumental [learning] trajectories**

The most recent notion I coined is that of **"instrumental learning trajectories**", which has been accepted by the "*International Institute for Science Technology and Education*" (Patsiomitou, 2021a).

I have introduced the notion of an *instrumental learning path/trajectory* (Patsiomitou, 2021a, b) as the interdependence/intra-dependence between dynamic tools, diagrams and mathematical objects during an instrumental decoding process. Instrumental trajectories can show us these dependencies that exist or can be created between different tools. Instrumental trajectories are not just construction instructions, or a set of information which provides the properties of the figure as the figure is constructed. I note the role of *instrumental decoding* in a static or dynamic environment, and how the competence of the participants (students – teacher) can influence/impact the holistic result of the learning process by creating *interdependencies/intra-dependencies* during the construction of *instrumental learning trajectories*. Moreover, I have pointed out that: (a) *The notion of LVAR is directly linked to the notion of instrumental decoding*; (b) *A dynamic diagram* expresses the interdependencies between dynamic objects; (c) *A dynamic section* expresses the intra-dependencies / interdependencies between dynamic diagrams and mathematical objects. This is also the core idea: the construction of *dynamic interdependencies* between dynamic objects (or hybrid dynamic objects) diagrams or sections lies at the heart of the construction of *instrumental learning paths*. An instrumental path/trajectory is considered complete when it connects the verbal description to the visual description.

For the history, I presented for the first time the notion of "instrumental learning trajectories" in my lecture via Webex (http://math.uoc.gr/el/seminars.html) to the postgraduate students of the TMEM of the University of Crete (http://math.uoc.gr/ark/patsiomitou. pdf) (24-3-2021) with the aim of applying it to the school textbooks of Middle-High School mathematics (i.e., I proposed a "**Dynamic Euclidean Geometry**", using pseudo -Toulmin diagrams). With this concept in mind, I wrote the book: "*Instrumental learning paths in Geogebra*" the year 2021. The book contains dynamic paths aimed which seek to build a conceptual understanding of, and mental connections between, concepts in algebra, geometry and calculus. The book is available online to teachers at every level of education, but also to students (https://www.academia.edu/46858220/). My aims in developing LVAR (Patsiomitou, 2010) were to:

- **enrich** the existing curriculum with DGS–based problems that are adaptations and extensions of existing static activities, tasks and real-world situations;

- **enrich** students' experiences with more effective presentation and interaction techniques better suited to the DGS environment than to other didactic materials;
- **trigger** students' actual cognitions in geometry as well as their aesthetic and digital sense; and
- **attract** students to solve DGS problems designed to develop their mathematical understanding, deductive reasoning and formal Euclidean proof, either individually or in an orchestrated classroom process.

In order to show the impact these tools can have on children's cognitive, mental and emotional development, I employed various digital tools, such as the eclass environment, for which I have created a number of online classes which have contributed significantly to the teaching of mathematics over the Internet. For my teaching, I used child-centered methods and differentiated teaching. I also made use of flipped learning, with appropriate use of verbal-visual-real-dynamic representations. My project "*Creativity and Skills in Mathematics*" (Patsiomitou, 2021d), describes how implementing my teaching using digital media has had a direct impact on the skill development of my students, by having them engage with innovative tasks that required them to apply mathematical concepts and connect them with art and the aesthetic perception of the world that surrounds us. This project (see also Patsiomitou, 2012c) is a result of a model, which acted as a prototype and influenced students' socio-cognitive learning (Bandura, 1977), so that they were led to self-efficacy. This article does not set out to write about this project, but as I have mentioned many times, this and other projects play a major role in students' socio-cognitive learning.

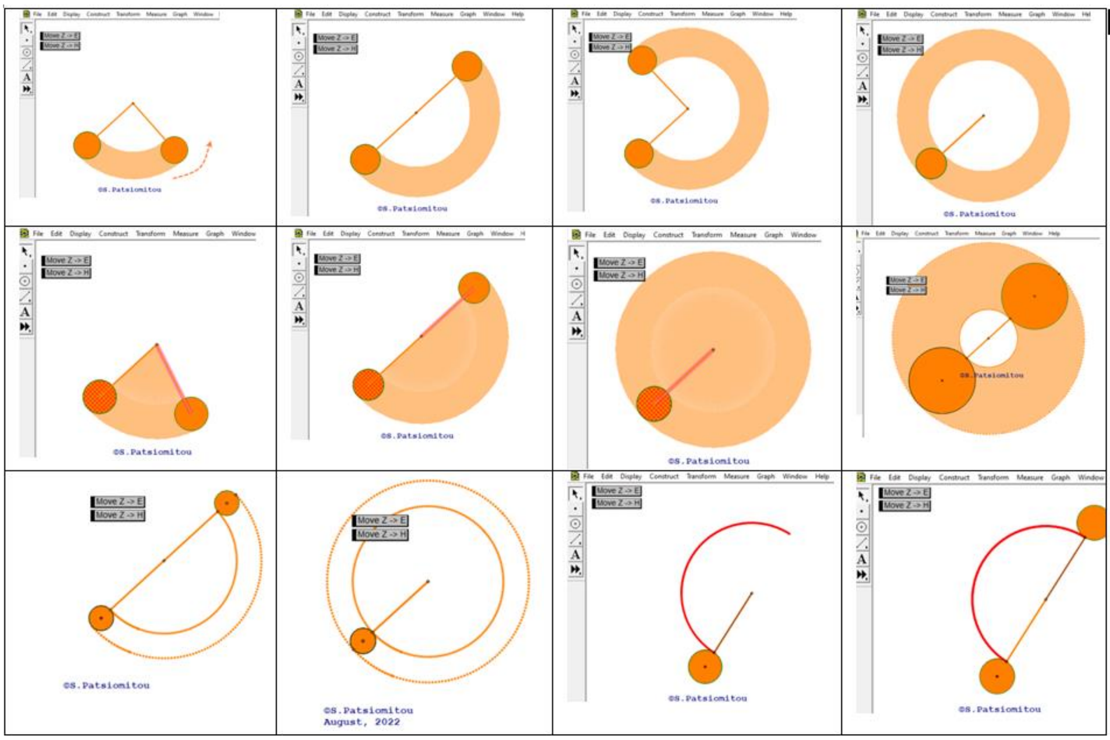


Fig. 13. The design of the first Fröbelian DGS-Gift using LVAR: screenshots of the ball's movement in the DGS environment with the traces thus produced (Patsiomitou, 2022d, p.8)

In year 2022, I also have published one more monograph in Greek (up to 470 pages): *Conceptual and Instrumental Trajectories using LVAR created with the Geometer's Sketchpad.*

I was honored that Prof. Daniel Scher, a leading expert in the use of The Geometer's Sketchpad and Web Sketchpad, undertook to write the preface of this book. The book is an upgrade version and evolution of my book "Learning *Mathematics with the Geometer's Sketchpad*" (Patsiomitou, 2009a, b). Both books consist of 15 stand-alone chapters on a common theme: the development of structures through sequential activities in Sketchpad which I designed using linking visual active representations. In the chapters of the book, I describe the way in which the included activities can be created, detailing the steps in their construction process and including illustrations. With the help of the concrete book, each teacher can create his own tools and activities by using the software's functions for the needs of his/her students. The book was approved by the *Greek Pedagogical Institute* and through the Greek Ministry of Education it had been sent to the libraries of the Experimental Model Secondary-level schools of Greece.

Recently, I have extended my research to other fields: primarily, in order to investigate LVAR use in primary education with young learners (e.g., Cuisenaire rods, Froebel Gifts, Dienes blocks, Montessori's activities, etc.). I shall continue my research to this field, as I believe we should begin with the mathematics that young learners understand and investigate how their geometrical thinking develops as a result of their interaction with linking visual active representations. And who knows? New definitions or terms may have to be introduced.

4. Instead of an epilogue: an objectification

Thurston (1994) insist that "*We mathematicians need to put far greater effort into communicating mathematical ideas. To accomplish this, we need to pay much more attention to communicating […] our ways of thinking*" (p. 168).

In writing the epilogue, I thought I would have an opportunity to discuss dynamic geometry in education: where it is now, where I anticipate it being a few years from now, and what role the concepts I coined and introduced will play in that evolution.

A point of reflection on the role dynamic geometry has played in education – and, at the same time, evidence on the impact it has on the thinking of students and users in general – can be got from Google and a scroll through the multitude of articles, dissertations and Ph.D. theses written about it around the world.

This evidence is overwhelming, as is the research that has been conducted into--and using--dynamic geometry software which describes its role in education in terms of the visualization and understanding of mathematical concepts, the development of students' levels of geometric thinking, and the development of proof/proving processes. The sheer amount of research conducted since 1990 has also helped to create a conceptualization around the field that most researchers now share. We are currently experiencing a period of significant change in education. I hope I am correct in assuming that no factors external to education itself will impact negatively on this development.

Schools are constantly changing: Blended teaching / learning has influenced the way we teach in school, but it has also impacted on the development of various activities which are mentioned in school textbooks for students and guides for teachers. Dynamic geometry software should and does play a crucial role in the teaching and learning of concepts in every field of science, at every level of education, from kindergarten to university.

Do all the teachers have shown the effects that DG environments have to their students' mind? Do they understand how important is the incorporation of these tools in every day lesson? Even if for their own cognitive improvement?

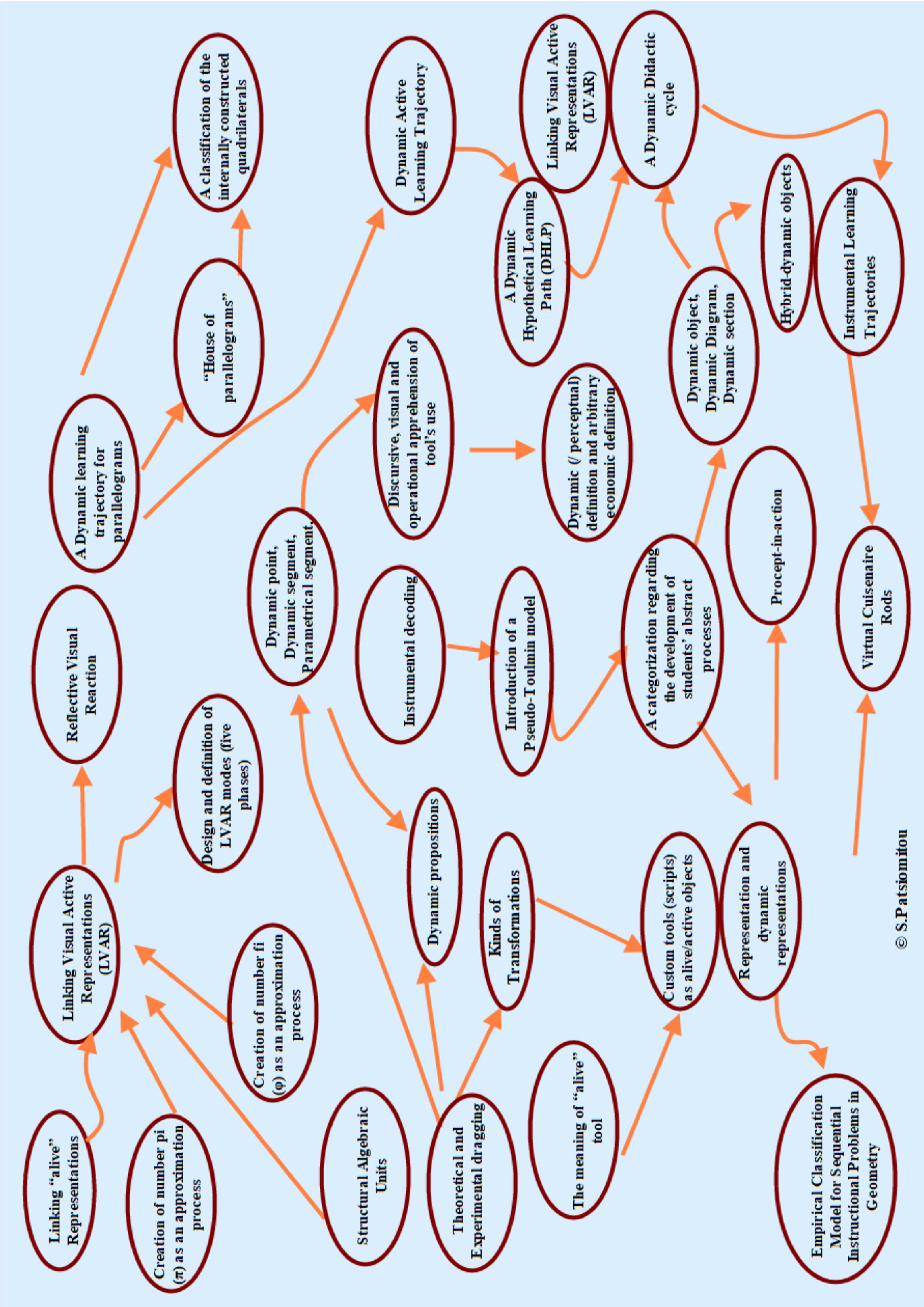


Fig. 14. A conceptual map of the dynamic notions listed in Table 1

Radford (2002) describes how van Gogh was able “to see what he was not able to see before”, after Mauve, his teacher, have shown it to him. I quote the passage from the letter van Gogh sent his brother (included in Radford, 2002, p. 14). “Mauve has taught me to see many things that I used not to see […or] you do not see properly either”. Radford then reports the excerpt as an example of “a process of objectification: […] a process aimed at bringing something in front of someone’s attention or view” (p.14).

I hope this article will become a process of objectification and bring these new terms, which can change the way we use, communicate, teach and learn mathematical ideas using DGS environments, to mathematicians’ attention.

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