International Journal of Scientific and Management Research
Volume 6 Issue 03 (March) 2023
ISSN: 2581-6888
Page: 18-40

Developing and Managing Knowledge through the Eyes of the Young Learner: 'Alive' Manipulatives before Abstract Notions

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Abstract

The current study has been influenced by that aspect of the philosophy of Realistic Mathematics Education which holds that mathematics should be learned as an activity of progressive mathematization. In this study I have used the Geometer's Sketchpad to design tasks for young learners, anticipating their reactions. My aim is to propose and introduce types of tasks /problems designed for primary-level learners, concentrating on two aspects: (a) using play-based "alive" tasks to demonstrate mathematical concepts (b) linking the sequential visual representations with instrumentally decoded math notions, thus making the whole figural diagram "alive", while giving the students the potential to focus their attention on simultaneous modifications (and transformations) of objects on the screen. In the current study, I paraphrase Pestalozzi's words "things before words, concrete before abstract" as "active or alive manipulatives before words, 'dynamic' concrete before abstract".

Keywords: Alive manipulatives, DGS environments, math concepts, RME, progressive mathematization

1. Introduction: Realistic Mathematics Education

Wubbels, Korthagen, & Broekman (1997) in their substantial article "Preparing teachers for realistic mathematics education" report the considerable change made by the characteristics of the philosophy of Realistic Mathematics Education (abbreviated as RME) thus "shifting from a mechanistic and sometimes structuralistic to a so-called 'realistic' approach, both in primary and secondary education" (p. 1). As they write (Wubbels et al., 1997, p. 2):

"Realistic mathematics education […] aims at the construction by children of their own mathematical knowledge by giving meaning to problems from a real-world context (Freudenthal, 1978; Treffers, 1987) […] realistic mathematics education does not start from abstract principles or rules with the aim to learn to apply these in concrete situations, nor does it focus on an instrumental type of knowledge".

If the students are engaged in investigating /or solving a real-world problem this process is underlined by RME, which "has been mostly determined by Freudenthal's (1977) view on mathematics […] Later on, Treffers (1978, 1987) explicitly formulated the idea of two types of mathematization in an educational context" (Van den Heuvel-Panhuizen, 2000, p.3).

The current paper is restricted to the aspect that mathematics should be learned as an activity of progressive mathematization, distinguished to horizontal mathematization and vertical mathematization (e.g., Treffers,1987; Van den Heuvel-Panhuizen, 1996; Gravemeijer, 1994; Drijvers, 2003). Horizontal mathematization in real world situations refers to the process of modeling from the real world to the model world using mathematical representations. In other words, horizontal mathematization is a process through which a real problem is transformed to a model. Vertical mathematization concerns the mathematical abstract process in a higher level of abstraction, connecting concepts and strategies. According to Freudenthal (1973) mathematics education should be a process of guided reinvention. The method of guided reinvention is linked epistemologically with the Socratic Method ("maieftiki" in Greek) by which teachers ask questions designed to elicit the correct answer and reasoning processes. The questioning process thus helps students determine and extend their underlying knowledge.

> *"Freudenthal saw the reinvention approach as an elaboration of the Socratic method and to illustrate the Socratic method, he spokes of 'thought experiments', i.e., the thought experiment of teachers or textbook authors who imagine they are teaching students while interacting with the man dealing with their probable reactions. One part of the thought-experiment, therefore, lies in anticipating student reactions. The other part consists in the design of a course of action that fits anticipated student reactions. More precisely, the idea is that teaching matter is re-invented by students in such interaction"*
> (Gravemeijer & Terwel, 2000, p. 786).

2. DGS environments and RME

Computers and especially dynamic geometry enviroments [e.g., The Geometer's Sketchpad, (Jackiw, 1991/2001), Cabri II (Laborde, Baulac, & Bellemain, 1988), Geogebra (Hohenwarter, 2001), Cinderella (Richter-Gebert & Kortenkamp, 1999), Cabri 3D (Laborde, 2004)], can be a source of meaningful problems with a variety of solution strategies. Students can explore the various solution paths individually and in small groups, making decisions and receiving feedback about their ideas and strategies. Another aspect of the flexibility provided by computers is the potentiality to enhance and reorganize information. The subject under analysis, are the students' interactions and the way in which they represent and verbally discuss their thoughts. Cobb, Yackel, & Wood (1992, p.4) suggest that "students construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations". Relative research has led to the identification of a series of characteristics of powerful learning environments (De Corte, 2000, p.254):

- *Learning environments should induce and support constructive, cumulative, and goal-oriented acquisition processes […] through a good balance between discovery learning and personal exploration on the one hand, and systematic instruction and guidance on the other.*
- *Learning environments should foster students' self-regulation of their learning processes […].*
- *Learning environments should embed acquisition processes as much as possible in authentic contexts that have personal meaning for students, […] and offer ample opportunities for collaboration.*

- *Learning environments should flexibly adapt the instructional support, […] taking into account individual differences […]*

Dynamic geometry systems (DGS) are learning environments, microworlds designed to facilitate the teaching and learning of Euclidean geometry, Algebra and Calculus. Microworlds have been described (Edwards, 1998, p. 74) "as 'embodiments' of mathematical or scientific ideas". As reported in de Villiers (1996): The development of dynamic geometry software in recent years is certainly the most exciting development in geometry since Euclid. Besides rekindling interest in some basic research in geometry, it has revitalized the teaching of geometry in many countries where Euclidean geometry was in danger of being thrown into the trashcan of history (p. 25). Many researchers argue that working in a dynamic geometry allows students to reinvent their personal knowledge by interacting with the other members of the group or with the teacher (or the participating researcher). For example, Furringhetti & Paola (2003) support that "in this case, the reinvention is guided, […] by the use of the [dynamic geometry] environment". Looked at from this point of view, learning geometry is a human activity and learning becomes a process of 'dynamic' reinvention (e.g., Patsiomitou, 2012a, b, 2014), following on from the guided reinvention posited by Freudenthal (1973). 'Dynamic' reinvention differs qualitatively from the Socratic Method because the aim of the method is the students to completely participate, undertaking active role by self-acting for the construction of meanings. Dynamic reinvention is thus an elaboration on the Socratic Method.

Hadas, Hershkowitz & Schwarz (2000) report, among others, that DGS lead students to be convinced that a conclusion is right, based on inductive trials (cited in Gorev, Gurevich, & Barabash, 2004, p.1). Arcavi & Hadas (1996) also outlined the following educational implications when we use DGS:

- *In the dynamic geometry environment, students are led to explore and to play with many particular cases. As a result, these observations may add insight and provide a basis for proving and further exploration.*
- *Students can conduct explorations and conjecture independently without the need for a teacher to confirm or judge the outcome. The role of the teacher can then be that of a guide that forces students to take a stance on conjectures and asks the all-important question "why?"*
- *Making sense of a situation while playing with the situation itself first enhances both the understanding of the situation and the representations used to analyze the situation such as measures, graphs, and symbols* (cited in Gillis, 2005, p.24).

Over the 18 years I have been using various software environments I have employed them for many different educational /instructional purposes and functions, which I have published (papers/articles or monographs in peer-reviewed international or panhellenic conferences and journals)(e.g., Patsiomitou, 2005a. b, 2006a, b, c, d, e, f, g, 2007a, b, c, d, e, 2008 a, b, c, d, e, f, g, h, 2009a,b,c,d,e,f,g, h, 2010, 2011a, b, 2012a, b, c, d, 2013a, b, 2014, 2015a, b, c, d, 2016a, b, c, 2018 a, b, 2019 a, b, c, 2020 a, b, c, d. 2021a, b, c, 2022 a, b, c, d, e). In my opinion, a DGS software can play a fruitful and crucial role in the process of creating and evaluating conjectures which promote student creativity, and in so doing greatly contribute to developing mathematical reasoning. In Resnick's (1998) words "*Learners make deeper cognitive connections when […] activities involve objects and actions that are familiar and relevant, learners can leverage their previous knowledge, connecting new ideas to previously-*

constructed mental models (p.46). My work with students at the secondary and tertiary levels led me to identify five types of geometrical problems (Patsiomitou, 2019a, p. 3): "(a) *Dynamic geometrical problems with non-given answers* (abbreviated as DGNA) which the students investigate in a DGS environment using linking visual active representations (LVARs) (e.g., Patsiomitou, 2008a, b, 2012a, b). Such problems improve motivation and creativity through the use of "why" challenges and "what if" strategies, while provoke students' reflecting visual reaction (RVR) (e.g., Patsiomitou, 2008a, b, 2012a, b) […]. (b) *Dynamic geometrical problems with given answers* (abbreviated as DGGA**)** which the students investigate and prove in a DGS environment. Such problems motivate students to create theoretical relationship between information and data which is explicitly provided […]. (c) *Dynamic geometrical problems modeled in a DGS with hybrid–dynamic geometrical representations* (Patsiomitou, 2019b, p.42) *with non-given answers* (abbreviated as HGNA) which the students investigate in a DGS environment. Such problems require the students to interact with a sophisticated level of information and data which is explicitly provided in the DGS environment […]. (d) *Real world geometrical problems with non-given answers* (abbreviated as RGNA) which students investigate in a dynamic or static environment. Such problems relate to 'dynamic' methods in geometry and require students to 'think in motion' in the environment […]. (e) *Static geometrical problems with given answers* (abbreviated as SGGA) which students solve in a paper-pencil or static environment […] applying their theoretical knowledge and perceive the structure of the problem and the principles and concepts that could be used to solve it".

In the current study, I used the Geometer's Sketchpad to design tasks for young learners, *anticipating their reactions*. My aim is to propose and introduce an adapted version of the types of tasks /problems that are posed to primary-level learners. The most important effect on students' thinking stems from the property of GSP to enable the designing of tasks/activities/dynamic problems by creating sequential linking pages. This has the effect of turning the whole Sketchpad file into an "*alive book*" (Patsiomitou, 2005a, p. 63, in Greek; Patsiomitou, 2014; Patsiomitou, 2018b, p. 40). The *"alive digital representations*" (Patsiomitou, 2005a, p. 67) make the whole figural diagram "*alive*", giving the students *the potential to focus their attention on simultaneous modifications (and transformations) of objects on the screen* (Patsiomitou, 2005a, p. 68), also yielded important results during my investigations.

3. Dynamic manipulatives as 'alive' representations

Manipulatives or concrete representations are objects (e.g., Cuisenaire rods, Dienes blocks) which are designed to mediate between a particular mathematical concept and the way pupils learn the concept. Pupils can manipulate them by touching or moving, and thus are concrete means (e.g., Dienes, 1960; Baroody, 1989; Van de Walle et al., 2005). Ross (2004) defines manipulatives as: "[…] *materials that represent explicitly and concretely mathematical ideas that are abstract. They have visual and tactile appeal and can be manipulated by pupils through hands-on experiences*" (p. 5). Generally speaking, a virtual manipulative is defined as "*an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge*" (Moyer et al., 2002, p. 373). In Pestalozzi's words: "*things before words, concrete before abstract*" (Pestalozzi, 1803 cited in Resnick, 1998). In the current study, these words are paraphrased as "*active or alive manipulatives before words, concrete before abstract*". Dienes (1960) originally postulated four principles of mathematical learning through which teachers /educators could create such tasks and activities.

- *The construction principle* suggests that reflective abstraction on physical and mental actions on concrete (manipulative) materials result in the formation of mathematical relations.
- *The multiple embodiment principle* posits that by varying the contexts, situations and frames in which isomorphic structures occur, the learner is presented opportunities via which structural (conceptual) mathematical similarities can be abstracted.
- *The Dynamic principle* states that transformations within one model correspond to transformations in an isomorphic model although the embodiments of these models are different.
- *The Perceptual variability* principle recommends that when presenting problem situations one should include perceptual distractors, i.e., one should vary the perceptual details of the problem but include some common structural characteristics so that students have an opportunity to link structurally similar problems.

The four principles of mathematical learning (Dienes, 1960, cited in Sriraman, & English, 2005, p. 258)

Researchers in the sphere of the Didactics of Mathematics take different approaches to conceptual determination, the theoretical interpretation of the notion of representation, and the ways that representations are used, investigating the teaching and learning of mathematical objects. The various interpretations of the terms associated with mathematics teaching and learning have been reported from many researchers always pinpointing the significant role that they have played for the development of students' mathematical meanings in simulations and problem solving (see for example, Presmeg, 1986; Janvier, 1987; Vergnaud, 1987; Lesh, Post & Behr,1987; Vinner, 1989; Dreyfus, 1991; Zimmermann & Cunningham, 1991; Kaput, 1987, 1991, 1999; diSessa, 1994; Duval, 1995; Aspinwall, 1995; Goldin, 1998; Hitt, 1998; Ainsworth, 1999, 2006, cited in Patsiomitou, 2019c, see also Chapter II).

Goldin (1998) in his study "*Representational Systems, Learning, and Problem Solving in Mathematics*" denotes the notion of "Representational systems" or "representational modes," as those systems "*which include systems of spoken symbols, written symbols, static figural models or pictures, manipulative models, and real-world situations, discussed by Lesh (1981*) […]" (p.143). He defines them as "*external systems of representation*". According to Goldin & Janvier (1998, p.1), these terms include the following: (a) *"An external,* structured physical situation, or structured set of situations in the physical environment, that can be described mathematically or seen as embodying mathematical ideas"; (b) *"A linguistic embodiment,* or a system of language, where a problem is posed or mathematics is discussed, with emphasis on syntactic and semantic structural characteristics"; (c) *"A formal mathematical construct,* or a system of constructs, that can represent situations through symbols or through a system of symbols, usually obeying certain axioms or conforming to precise definitions--including mathematical constructs that may represent aspects of other mathematical constructs"; (d) *"An internal, individual cognitive configuration,* or a complex system of such configurations, inferred from behavior or introspection, describing some aspects of the processes of mathematical thinking and problem solving". A number of studies have reported on external and internal representations, distinguishing between a representation of the meaning as an artefact and a representation as a mental construct (Goldin & Shteingold, 2001). Applying the constructivist viewpoint of Cobb, Yackel, and Wood (1992) external representations occur in the students' environment, while internal representations are developed in the student's thought. Thus "*cognitive development is closely related to the ability to represented objects* *states of affairs and relations, either internally or externally and to move successfully between the two*" (Sakonidis, 1994, p.39).

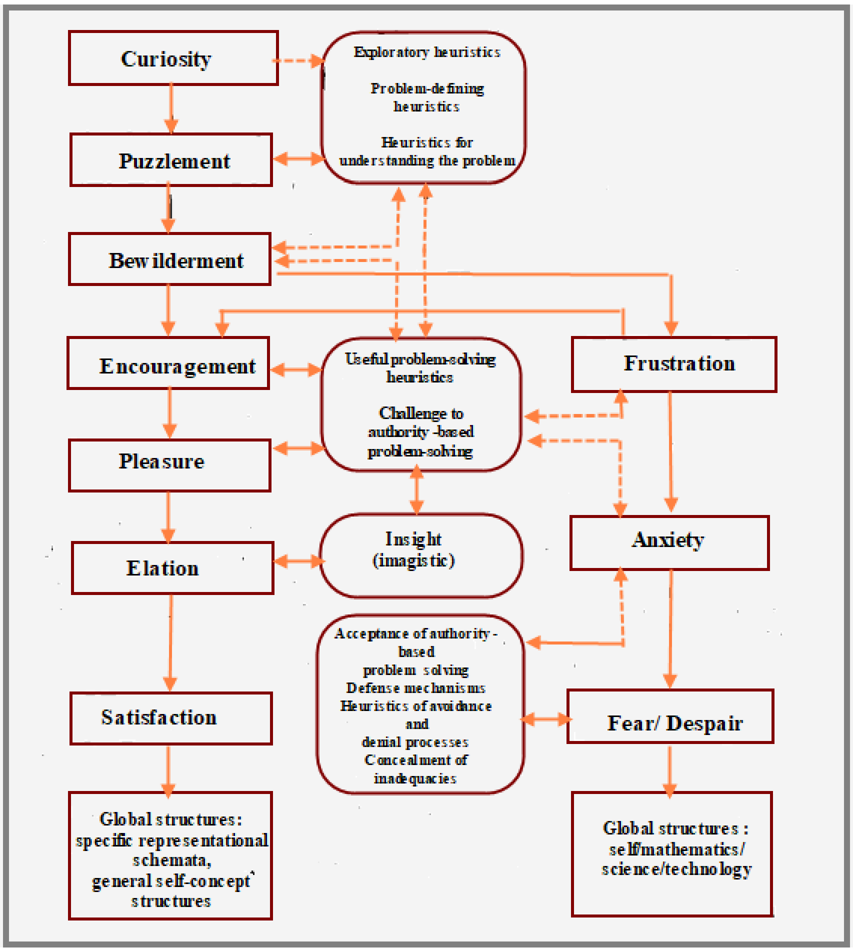

Figure 1. Affective states interacting with heuristic configurations (Goldin, 2000, p. 213) (an adaptation for the current study, see also Patsiomitou, 2019c, chapter II)

This latter point is of crucial importance as students' external constructs can serve as an indication of their internal constructs (Kaput, 1989), and simultaneously an indication of their understanding. Goldin has elaborated on the role that affective states play in the problem–solving process in numerous articles. Goldin (2000) in his study "*Affective Pathways and Representation in Mathematical Problem Solving*" constructed a realistic model from problem-solving competence. He outlined in the figure above (Figure 1) and discussed in the article, "*two major affective pathways, one favorable and one unfavorable,*" […] "*Thus affect, like language, is seen as fundamentally representational as well as communicative […]it cannot be handled simply by a commitment to make mathematics fun or enjoyable") (p. 209).*

Having taken on board the definitions of the term "representation" in the literature (see also Patsiomitou, 2019c, chapter II), I think that a representation is both (a*) an external entity (such as a verbal expression, a graph, a figure, a map, a picture), which is to say an external correspondence of objects or processes with the objects that are represented by the entities brought into being as representing objects by the modelling process, and (b) an internal mental entity, meaning a structurally equivalent modification of physical/mental objects/processes which are constructed in the mind as a result of the processing/elaboration of information and* *the manipulation of objects and concepts due to the cognitive schemes which have developed in the subject's mind* (see also, Patsiomitou, 2019c, p. 42, in Chapter II).

4. The design of the tasks: 'Alive' objects before words

The current paper is not concerned with a detailed description of the design process, and will instead concentrate on two of its aspects: (a) using play-based tasks to demonstrate mathematical notions (b) linking the sequential dynamic linking visual 'alive' representations with math concepts, *instrumentally decoded* (Patsiomitou, 2011a, b) by different tools. Moreover, I tried to f*acilitate connections leveraging the intuitions and interests that children have developed from their lifelong interactions and experiences in the physical world* (Resnick, 1998), taking into account what Figueiredo, van Galen, & Gravemeijer (2009) support:

> *The problem with designing for mathematics education is that the designer already knows the mathematical concepts, which are the goals of the tasks. Designers have to forget this, in a sense, and put themselves in the shoes of the learner. This asks for a shift in perspective, which is called a shift from an observer's point of view to an actor's point of view (Gravemeijer, 2004; Cobb & Bowers, 1999; Cobb, 1987)*

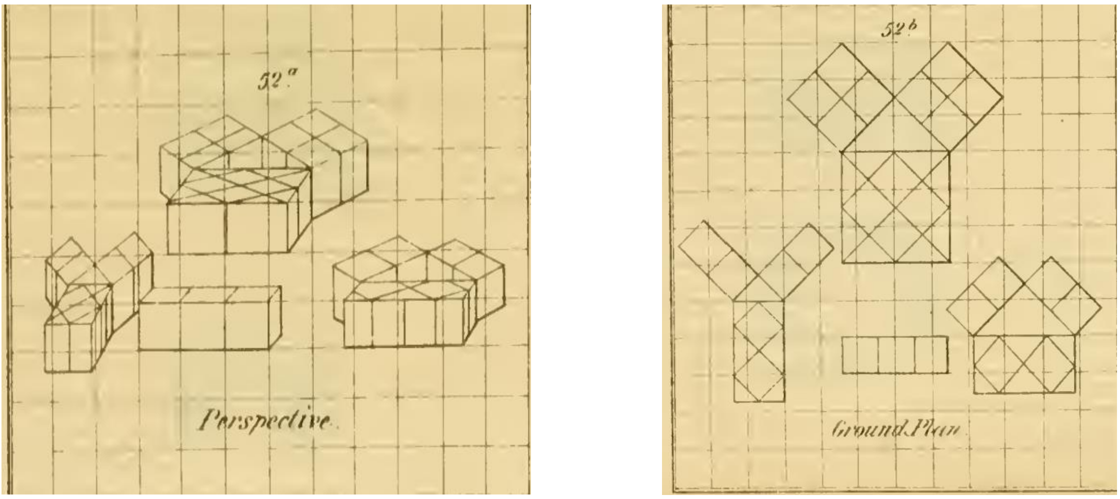

Figure 2a, b. A task mentioned in Froebel's fifth gift (adapted) (Ronge, & Ronge, 1858, p.105)

The figures above (Figure 2a, b) present different perspectives of the same construction: how we can view the cubes on the ground and from above. Froebel's fifth gift "*consists in the increased number of parts, by which more extended operations can be carried on; and the introduction of triangular forms, by which new forms of beauty can be produced, a greater variety of buildings* […]" (Ronge, & Ronge, 1858, p.32). The figures in Figure 2a appear like a 3D version of the Pythagorean theorem: for example, the figure in the middle consists of 4cubes + 4 cubes which are equal to the 8 cubes that create the square of the hypotenuse.

It is very easy to create a metaphor for this task in a DGS environment. How useful is this for young learners? I absolutely agree with Resnick's (1998) words that "*most applications of computers in education, use computers in rather superficial ways. They take traditional classroom activities and simply reimplement them on the computer. The activities might be somewhat more engaging, and the computer might provide some additional feedback, but the activities themselves are not changed in fundamental ways*" (p.3).

On the other hand, what is the qualitative difference between the figures 3, 4a, b? The verbal formulation of the task is the following *(Figure 3, 4 a, b)*: *If we add the odd numbers 1, 3, 5, 7, 9, 11 ..they form the square numbers 1, 4, 9, 16…., meaning 1+3=4, 4+5=9, 9+7=16……(*Ronge, & Ronge, 1858, p. 28). On the left, Figure 3 demonstrates cubes that form the square numbers. On the right, there are two demonstrations that represent what is reported in the verbal formulation below the images, giving the pupils the opportunity to develop their *spatial thinking*. In other words, the adapted DGS version of the Froebel's Fourth gift (Ronge, & Ronge, 1858, p. 28) differs qualitatively from the static version.

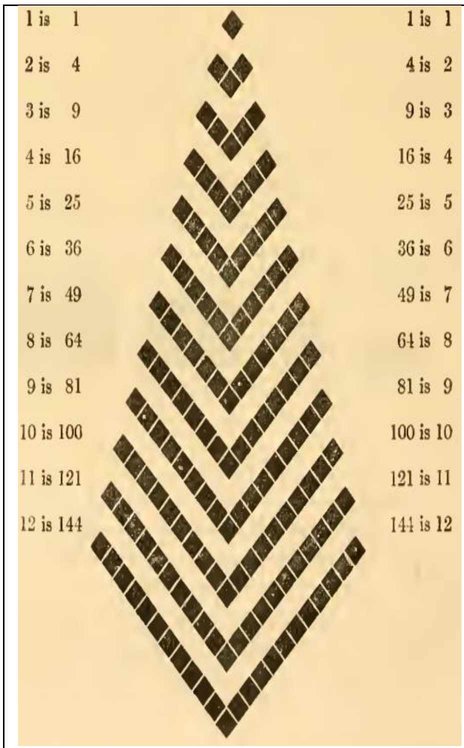

Figure 3. A task mentioned in Froebel's Fourth gift (Ronge, & Ronge, 1858, p. 28)

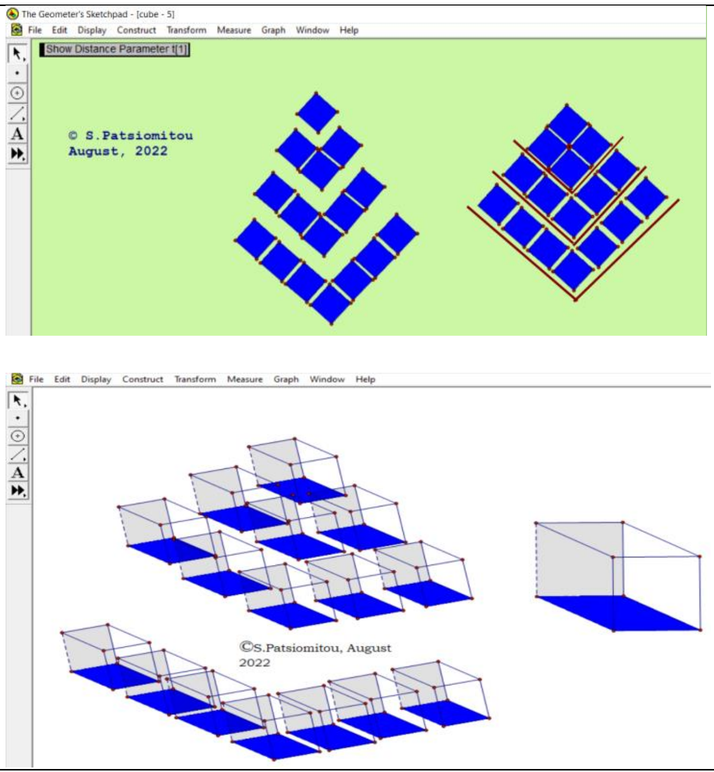

Figure 4a, b. The adapted vesrion of the task in GSP

Many *researchers have* discussed the relation between visuo-spatial thinking and mathematics (e.g., Bishop, 1986; Clements & Battista, 1992). *Spatial thinking* is the kind of thinking that enables the students' understanding of objects' location, size and orientation. Moreover, the visuo-spatial thinking that develops during experimentation in a DGS environment allows pupils to visualize the spatial aspects of different related objects. Spatial thinking also, allows young learners to imagine 3D objects and visualize or manipulate them in their mind. Pupils can transform the cubes and connect the iconic representation with the verbal formulation (the "theorem") in their mind. Specifically, the process of demonstrating an activity or a problem is made up of a series of steps which can function as a response, anticipating the questions posed explicitly by the teacher (or implicitly by the student). The completion of a step could correspond to a different page in the software connected to the previous page via link action buttons, while on every page different constructional or exploratory actions are linked with a mathematical notion seeing from a different perspective. The dynamic linking representations that are demonstrated in the sequential pages more complex and based on the previous (see also, Patsiomitou, 2008a). The students can be led to conclusions which compose a step-by-step visual evolution of an explanation of the activity/task or problem. The resulting interlinked successive pages could be compared with an "alive" section of a textbook (Patsiomitou, 2005a, p.20).

The design of activities in a learning environment (the software) as a part of the instruction thus has a crucial role to play in the comprehension of mathematical meanings and the acquisition of a higher level. In this way, the sequence of increasingly sophisticated construction steps in the activity could correspond to the sequence of the linking tools, allowing the student to interact with the tool [on his/her own volition or on being encouraged to do so by the teacher during the time that the activity is demonstrated]. The link tools (i.e., action-buttons, link-buttons) can join different pages and later, more evolved steps in the construction of the representation of the problem. This leads to a cognitive linking of the representations which (Kaput, 1989) *"creates a whole that is more than the sum of its parts...It enables us to see complex ideas in a new way and apply them more effectively*". In that way the pupils can improve their knowledge by having a mental schema elicited from them. In the next section I am presenting 5 tasks that I created in sequential pages of the Geometer's Sketchpad DGS environment.

5. The tasks

Task 1st: Coloring diagonals in Dienes' block (a DGNA task)

The young learners learn how to represent numbers using coloured manipulatives (e.g., Cuisenaire rods, abacuses, Dienes cubes, Montessori colour beads, fraction circles, Geoboards, pattern blocks).

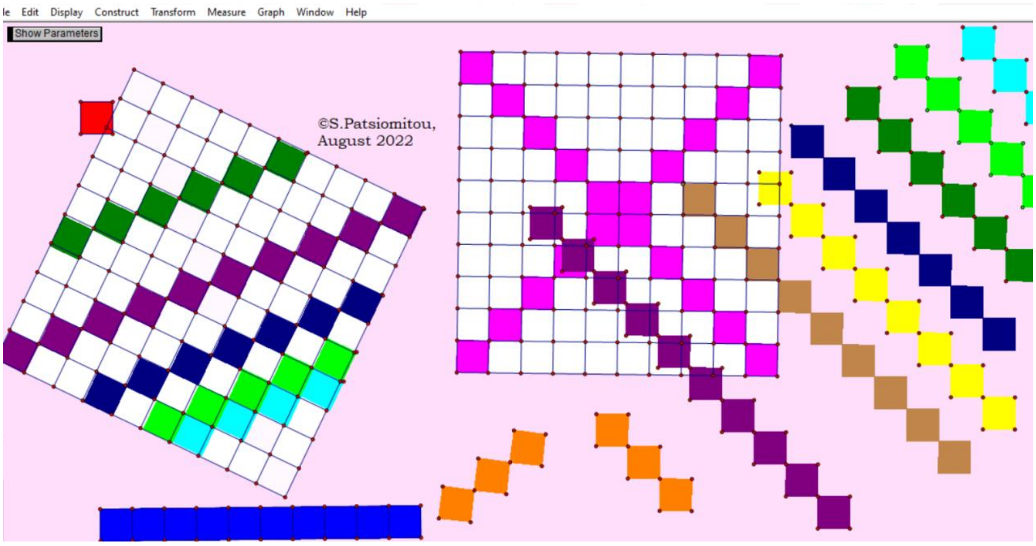

Figure 5. Combining Dienes blocks and DGS Cui-Rods

In terms of the current study, special attention was given to the construction of abstract units from shapes which represent the DGS manipulatives (see also Patsiomitou, 2022 b, c, d). Combining Dienes blocks and DGS Cui-Rods in elementary mathematics, children could learn geometrical concepts and develop their spatial sense as they interact with these activities. They are able to reinvent relationships among mathematical objects (e.g., shapes and angles) as they explore these activities and tasks. The coloring of the diagonal squares led me to another thought: could the students place the diagonal bars made up of 2, 3, 4 ...squares with the same ease into the boxes of the bigger square? Changing the orientation of the 10x10 block would also create obstacles for the students.

Task 2nd: Creating successive ladders (a DGGA task)

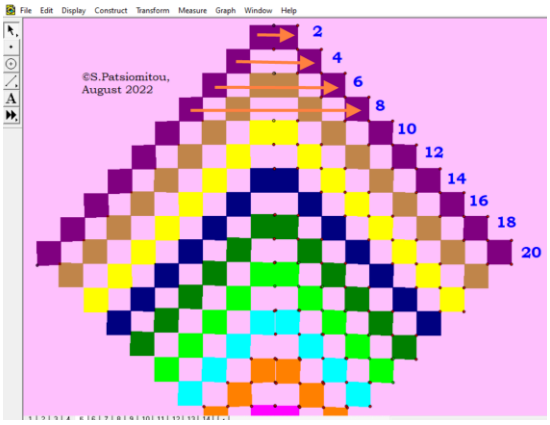

Figure 6. Combining Dienes blocks and DGS Cui-Rods

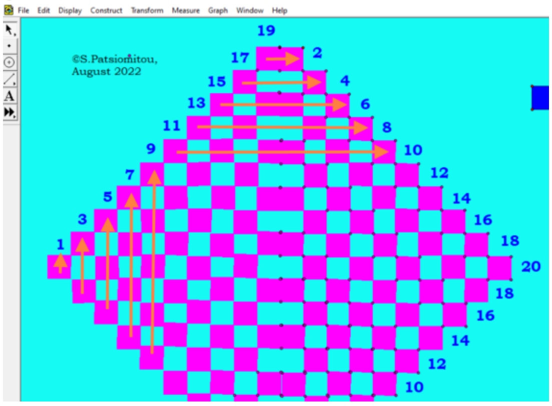

Figure 7. Combining Dienes blocks and DGS Cui-Rods

Placing the diagonal bars so that they form successive ladders can result in even numbers (see arrows for numbers 2, 4, 6 etc.) forming when the students count square boxes horizontally and odd numbers forming when they count them vertically (bottom image). This means we can count an even number of squares in the rows but an odd number of squares in the columns. Asking young students questions like how many-colored squares there are in total, or how many uncolored squares there are in the big square, can help them discover a pattern, a symmetrical property, or other properties they will express in an informal way.

Task 3rd: Doubling a square consisting of tans (a SGGA task)

Tangram is a very good task for all ages. At a young age, they easily compose the square from the puzzle pieces (static/physical manipulatives or dynamic: the tans), but older students can calculate the length of the segments/sides and the area of each figural piece. But can they think of a way to double the area of a squared tangram? When I presented the problem to my 13-year-old students, they doubled the length of the sides, but measuring the areas of the tans and the total square showed that the figures' areas had quadrupled.

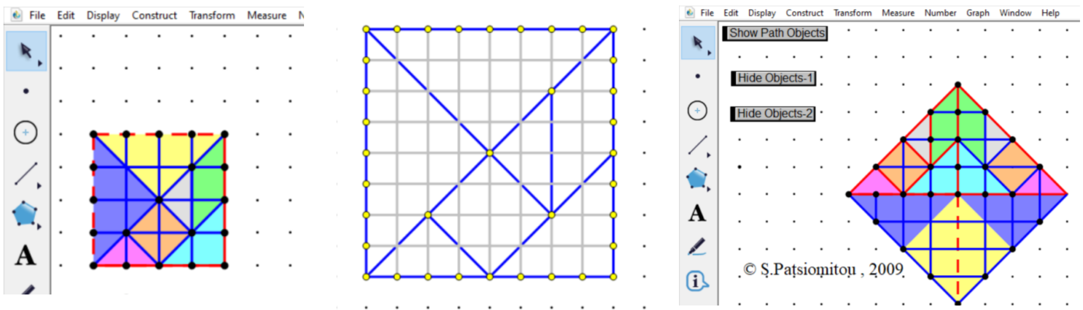

Figure 8 a, b, c. Doubling a square consisting of tans

Successive transformations of the pieces by reflection and rotation to form Figure 8c produces a tangram with twice the area and consisting of twice as many tan pieces. This task could also work for young learners, if we give them twice the number of pieces which, with suitable transformations, they can place to form a square tangram with double the area. "*For what is important when we give children a theorem to use is not that they should memorize it. […] One* *learns to enjoy and to respect the power of powerful ideas. One learns that the most powerful idea of all is the idea of powerful ideas*" (Papert, 1980, p.76).

The idea of creating a project based on the game led me to make up an open problem: *Lou has 14 tans pieces. Using 7 of them, she made a square consisting of all the tans. But how will he make a square with all 14 pieces? What is the area of the square now?* It is interesting for young learners to answer the following question: How many ways are there to construct a square like that from 14 tans?

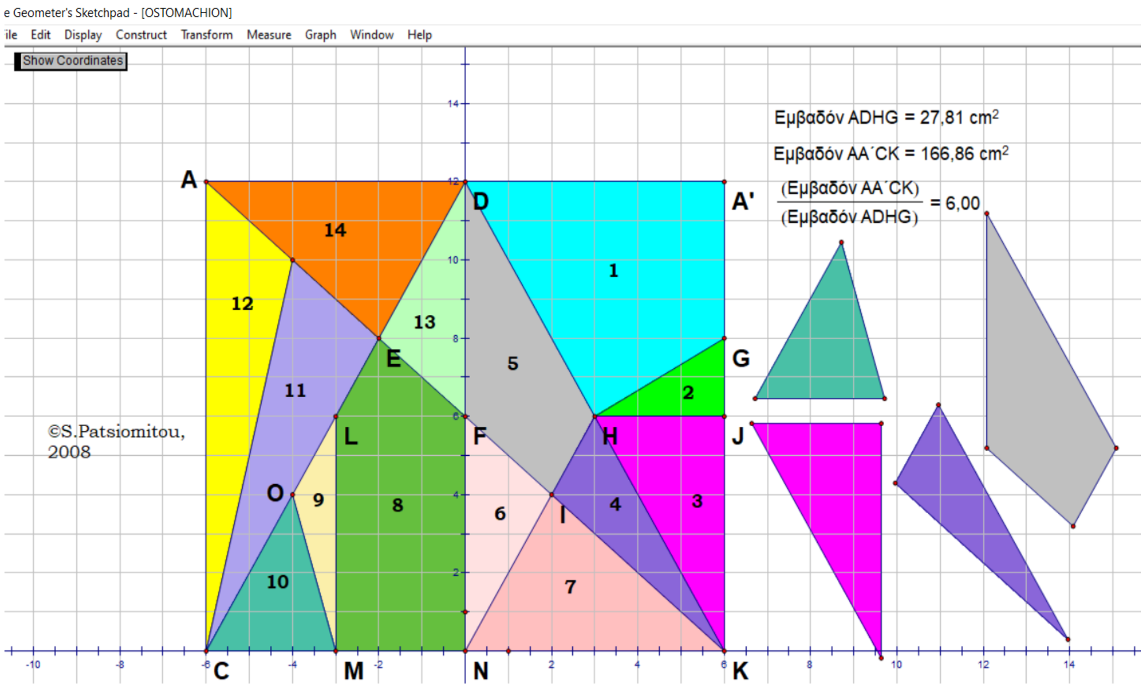

Figure 9. Puzzle pieces of the game Ostomachion, created in GSP (Patsiomitou, 2009a, b, 2022a)

The puzzle pieces of a game known in ancient Greece whose origins go back to Archimedes work in the same way. Ostomachion (a Greek word), also known as *loculus Archimedius* is a mathematical treatise attributed to Archimedes. (https://en.wikipedia.org/wiki/Ostomachion).

Task 4rth: A 3D game (a RGNA task)

"Building with blocks" is a math applet provided by the Freudenthal Institute for Science and Mathematics Education (FI). It is available from the Institute's website [http://www.fisme.science.uu.nl/toepassingen/28432/]. Students of any age can use this applet to play and develop their spatial reasoning. Parts of the diagram are hidden, but the student can change the orientation of the diagram to better view another option. Students can also add or remove blocks to "build" a construction (e.g., a castle).

In the figures 10, 11 below I designed colored cubes located on a rectangular grid, to enable students to visualize 3D objects and to predict the location of the next cube. A cube fits exactly on each of the indicated squares in the grid. In other words, teachers can ask pupils to count how many cubes it would take to completely cover the grid, if they followed the structure that the first two cubes have on the grid. Another task could be pupils to draw where they think or predict the cubes would be located on rectangular dot-paper (or squared paper) [that requires spatial reasoning skills].

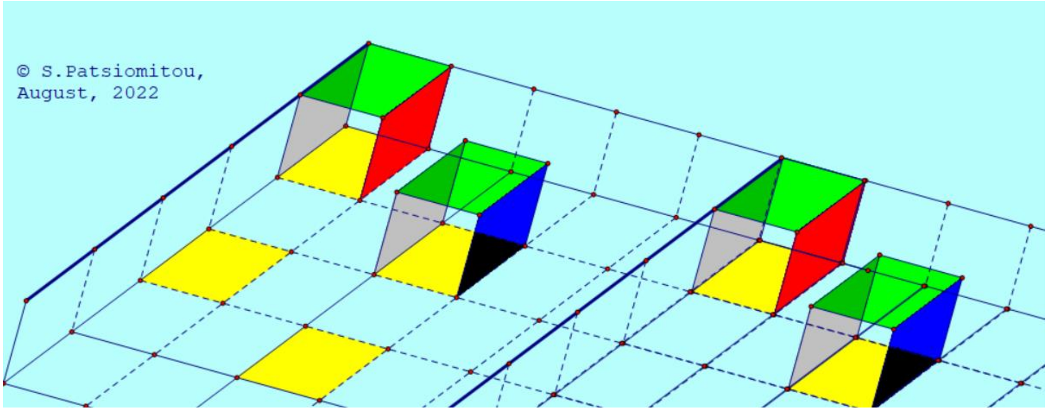

Figure 10. Colored cubes located on a GSP rectangular grid

It would be very interesting for the teacher to investigate the following research questions (See also, Gutiérrez, 1996; Battista et al., 1998; Christou et al., 2006):

- Can the pupils determine the number of cubes needed to cover the grid?
- Can they make sense of the way the first two cubes have structured the diagram?
- Can the pupils perceive the spatial relationships, meaning have the pupils the ability to relate several objects on the rectangular grid?

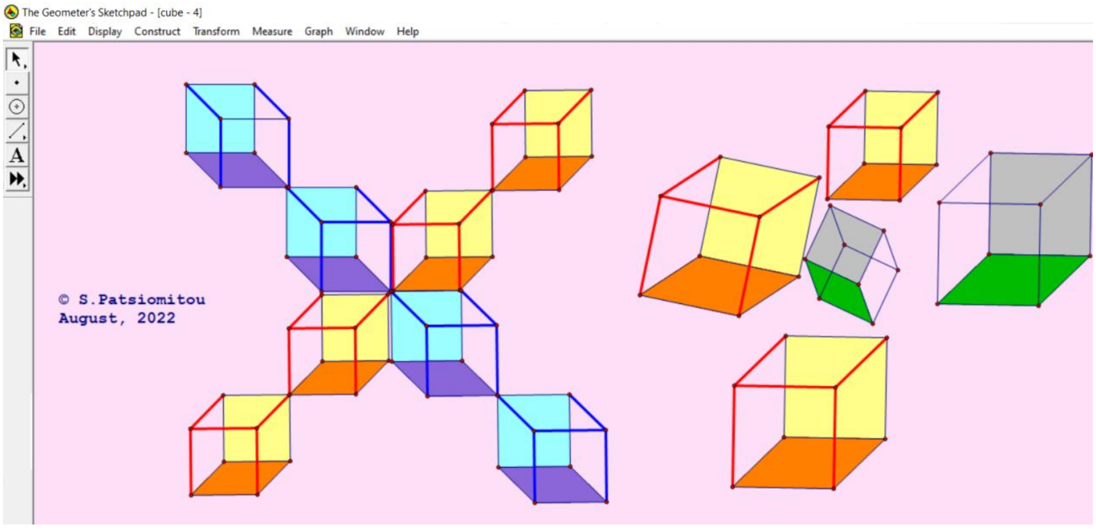

Figure 11. Colored cubes in a play-based DGS task

In the figure 11, the 3D diagram allows pupils to observe the same cubes in different positions. The teacher can resize all the objects by dragging, allowing students to visualize, investigate and experiment with 3D geometrical objects in the DGS environment. Moreover, in the figure 11 the DGS environment enables students to construct and manipulate geometrical 3D cubes in space, playing with them so that they form successive ladders. In my opinion, a pupil can develop his/her level of spatial ability by proceeding through increasingly complex, integrated figures and visualizations to a more complex linked representation of a task or a problem, and thereby moving instantaneously by means of mental consideration.

Task 5th: A real world problem with "alive" objects (a HGNA task)

In the following task, I have tried to make knowledge of the physical world more accessible to children. In the GSP file, I have created a simulation of a leaf (Figures 12a, b) as the resultant of a circular arc and its reflection (Patsiomitou, 2022d, p.10). By using parameters to scale it, I have reproduced it. We can also move these leaves away from one another or bring one leaf closer to another. Such observations can help a child develop strong experience in developing mathematical concepts and, subsequently, the schemata relating to those concepts. For example: a leaf consists of two circular arcs, the leaves have an axis of symmetry, an angle is formed between the branches of two leaves which is not always the same in all branches, etc. According to Freudenthal (1973) "*Geometry can only be meaningful if it exploits the relation of geometry to the experienced space. If the educator shirks this duty, he throws away an irretrievable chance. Geometry is one of the best opportunities that exist to learn how to mathematize reality*". (pp. 406-407). The modeled tasks are presented in the DGS environment. The pupils will be motivated to manipulate the dynamic objects and to compare them with the real-world objects, physically or in their minds.

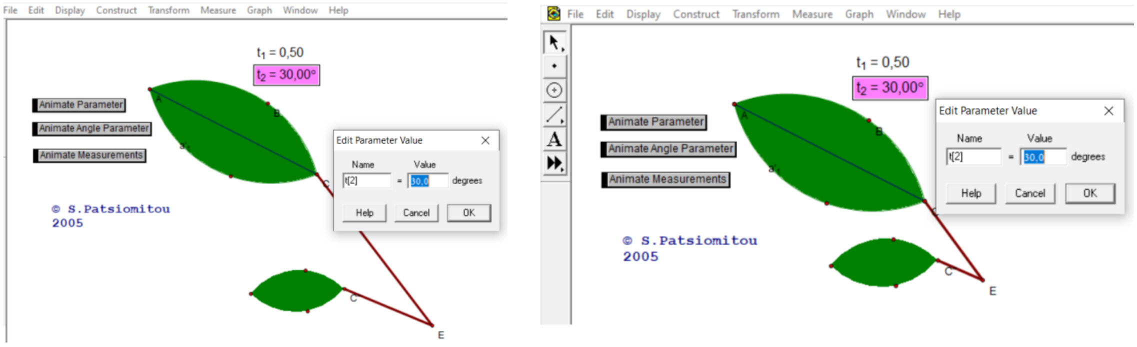

Figure 12 a, b. Playing with "leaves" in GSP (Patsiomitou, 2022d, p.10)

Properly designed transformations do not only transform the overall plan-structure, they also change the design of the bigger or smaller leaves.

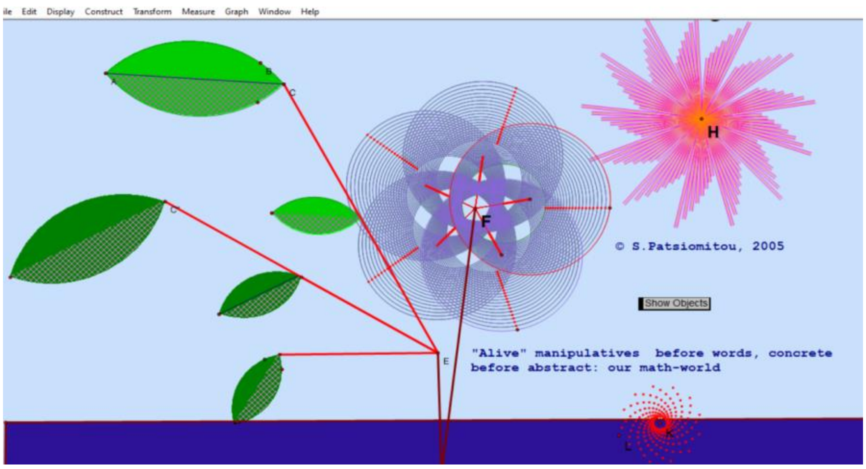

Figure 13a. Real world "mathetic" tasks

Children can recognize and describe the parts of these "alive" objects and also examine the parts of flowers: the stem, petals and leaves. In a discussion, pupils can compare the colours of the real flowers with the "alive" artefacts on the computer screen. They have green leaves which are symmetrical and similar to each other. The teacher could also prompt pupils to examine the similarities and differences between the real flower and the "alive" flowers on screen. Here is provided a new problem in a story: Lou, Thea and Alex (: young learners) went to play in the garden. Lina created an object in the DGS environment in which she tried to adapt what she had seen in the garden. Then they talked about it. What follows is a hypothetical discussion between Alex, Lou and Thea who are experimenting and playing with the objects in the DGS environment:

Lou: *These are leaves…the flower is in the water.*
Thea: *There are small leaves and big leaves.*
Thea: *The leaves have two parts*.
Lou: *The computer also drew a flower.*
Lina: *How many petals does the flower have?*
Thea: *It has five… But it's too big! It's a giant flower!*
Lou: *Can we make a smaller flower?*
Thea: *Here is the sun!*
Alex: *Can we make more leaves… we have to construct quarter circles*.
Lou: *We need five circles for the flower.*

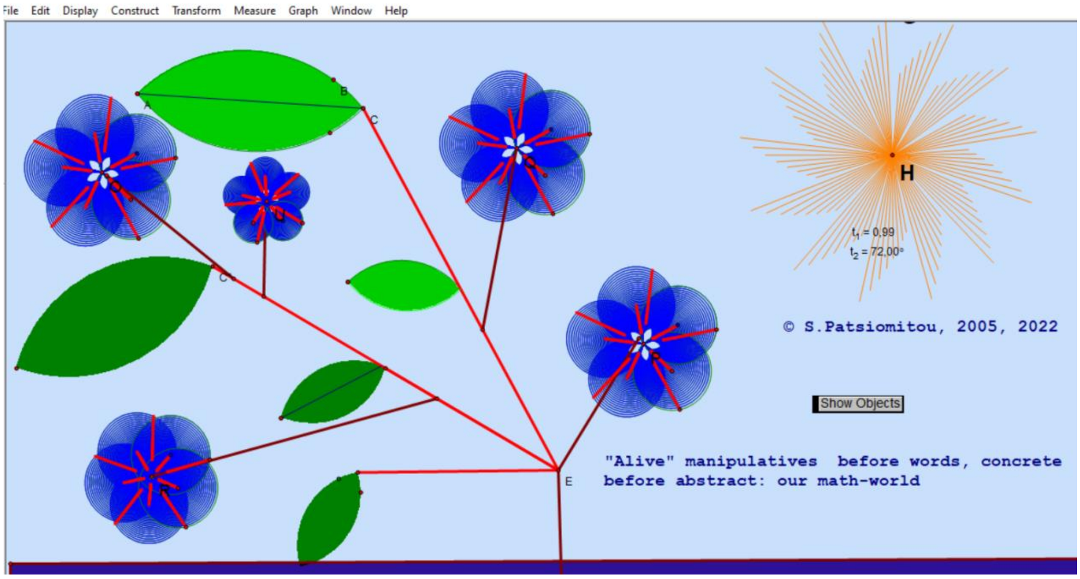

Figure 13b. Real world "mathetic" tasks

It is clear that I have followed Papert (1980) here, who introduces a dynamic proposition "*The Total Turtle Trip Theorem*" and uses this theorem in a hypothetical conversation to help/enable children to construct first a petal, then a flower, a new flower and, finally, a plant.

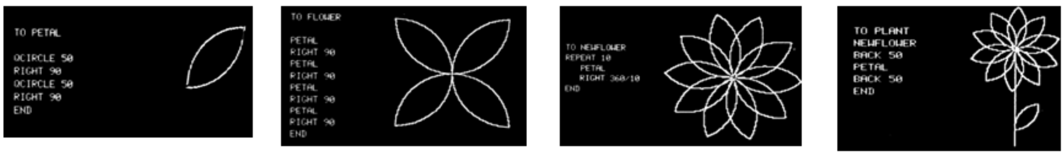

Figure 14a, b, c, d. From petal to plant (Papert, 1980, p.76)

Papert introduced the word "mathetic" to define another kind of knowledge: "*knowledge about learning*". He and his colleagues "introduced Turtle geometry by relating it to a fundamental mathetic principle: *Make sense of what you want to learn*". The mathematical experiences in the current study are theoretically based on dynamic representations. For instance, pupils construct the concept of flower with all possible modes of representations (i.e., verbal, figurative, dynamic). They start to conceptualize the geometric parts of the flower through links to associated representations. As Ronge & Ronge, 1858, write:

> *The importance of this exercise does not end with the power to copy natural or artificial objects. It has a moral effect; it leads the mind to observe the wonderful harmony existing in nature, and creates a longing after the source of so much beauty. Colour being the result of the influence of light; and light being an essential source of life, these exercises may lead naturally to the true source of light and life—to God. Children thus trained are not so much pleased with a mass of colours as they are with their harmony: and their constant effort is, whenever they have the opportunity, to collect the greatest variety of flowers, and arrange them according to their own ideal of the beautiful (*p.55).

5. Discussion

This study is part of a larger design-research effort I have undertaken on the issue of how task and activity design can help pupils experience and discover mathematical structures. The research questions under investigation relate to teaching/learning from tasks designed in the DGS:

- How do students' discussions during experimentation in DGS environments strengthen and impact on the links between procedural competency and conceptual understanding in mathematics?
- How can teachers design tasks in DGS environments which strengthen/influence/impact on student peer interaction which supports their learning of mathematics?
- What features of the design of tasks that utilize DGS environments will enhance students' initial observations/visualizations and perceptions/conceptions thereby promoting understanding and building new knowledge?

My further aim is to inculcate in teachers of mathematics a greater awareness of the theory and research into the Didactics of Mathematics, taking into account the impact representational technological environments have had on mathematics learning and teaching. As part of the leaning process, we have to understand what the mathematical objects are, how to use them and how to represent them in static or dynamic means. Language also plays a crucial role in the teaching and learning process. Do we learn alone as individuals, or with others in a social context? Do we learn using traditional means, or through e-learning and computer software? Both are important for students. As teachers, we have to choose the road, the learning path our students will follow, by using a *thought experiment* to construct a hypothetical dynamic learning path that predicts their progress and their thought development.

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