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Sensitivity and Joint Investment Analyses of a Multi-objective Investment Optimization Model of two Businesses in Ghana

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Abstract

This paper is a further development of an earlier work reported in the literature by the authors. In that work, the investment problems of two businesses in Ghana were formulated as bi-objective optimization models in which their expected returns and risks were optimized simultaneously using real data from the two. The current work investigates the two models under a sensitivity analysis and about the potential benefits that a joint investment could hold for the two businesses. The rates of return parameters were identified as the most likely to vary over the period of the investments. Therefore, they were varied up and down by 5% and up and down by 10%, considered as reasonable levels of variation. A random selection procedure was employed to obtain two random sets of rates of return parameter values for the two businesses, from which expected return and risk values were obtained for each of the models. The results of running the models (in MATLAB) showed that the impact of parameter variations was not significant, as far as the Pareto optimal solutions which are the amounts to invest, were concerned; the objective function values of returns and variances varied however, since their coefficient values varied with the values of the parameters varied. The formulated joint model produced results that showed that a joint investment could be more profitable than separate investments, even though that had a much higher risk.

Keywords: Sensitivity analysis, Variation of parameters, Joint modeling, Pareto optimal solutions.

1. Introduction

This work is a sequel to an earlier one by Akanyare & Twum (2022) which reported a bi-objective approach to optimizing the investment portfolio of two businesses in Ghana. Specifically, the reported work was focused on finding the amounts of money to allocate to several available investment portfolios in order to maximize the overall return on the investments, while at the same time minimizing the risk of the investments over the given period (Qu et al, 2017). As a very important decision-making task, investors and portfolio managers alike desire the most efficient and optimal strategy that inures to their benefit and ensures the growth and sustainability of their businesses. The current work assesses the two optimization models developed by Akanyare & Twum (2022) in a post-optimality analysis, to find out the impact of selected parameter variations on the solutions of the models and to investigate the prospects of a joint investment operation for the two businesses.

Post-optimality analysis which is characterized by sensitivity analysis of a model is an important part of model solution and comprehension (Taha, 2011), in view of the fact that model parameters are usually estimates or approximated values and, therefore, useful to assess the impact of slight variations of their values on the solution of the model. This enables assessment of the range of possibilities for the solution of the model and provides reasonable basis for decision making (Li et al., 2016). Sensitivity analysis may be seen as a qualitative analysis that studies the derivatives of the perturbed model (Tanino, 2014). It can be performed on any of the categories of parameters of the model or on a combination of the parameters, such as the coefficients of the objective functions, or those of the constraints, or even the resources utilization limit coefficients, without changing the essence of the problem. Besides, optimization problems under uncertain conditions abound in many real-life applications (Gugat et al., 2021) and so a post optimality analysis gives an idea about the likely behavior or response of a model to some amount of uncertainty. Sensitivity analysis therefore may be considered also as a practical strategy that reveals which parameters have significant or insignificant effects on the decision variables and objective functions (Li et al., 2016), and to what extent. It is useful for selecting the final solution in multiple objective problems and is very much applicable across all aspects of life (Avila et al., 2006). In crafting a good model, sensitivity analysis helps to verify that the inputs to a model conform to theory and may be useful in the model's calibration process (Saltelli, 1999). It is important to note that many authors mistake this for uncertainty analysis (Saltelli et al., 2019) whereas in reality, the two are not the same.

Joint modeling refers to the art or science of merging two or more models into a single model to function as a single unit. Literature and some experimental works have shown that joint models can outperform their individual separate models. For example Twum (2013) in his investigation, using Linear Programming, of prospects for a joint venture between two otherwise competing Firms in Ghana, found that the two stood to profit more in such a venture. Zamani & Croft, (2020) merged a retrieval model and a recommended model in neural systems and established that the joint model outperformed the individual systems. Thus, joint modeling can be taken up in any domain of modeling. In the work reported in this paper, a joint investment model of the two individual businesses studied in Akanyare & Twum (2022) will be formulated and assessed to determine whether or not there is profit in operating together.

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It is an established fact that decisions about investing in terms of where to invest or how much to invest or both cannot be taken lightly, and how one approaches the matter goes a long way in whether or not the endeavor is rewarding. This fact was realized long ago by Harry Markowitz, leading to his seminal work in the subject area of portfolio optimization (Markowitz, 1952). Indeed, Markowitz's ground-breaking work today forms the foundation of modern Portfolio theory. He provided a mathematical model to describe the impact of Portfolio diversification by the number of securities within the Portfolio and their covariance relationship. Markowitz states that, the expected return (Mean) and the risk (Variance or standard deviation of the expected return) of investments are the main criteria for portfolio selection and construction (Markowitz, 1959). For more examples of related works in this study area, see Qu et al (2017), Pandey (2012), Kamil & Kwan (2004), Wagner (2002), and Miettinen & Mäkelä (2002). Despite the fact that the Markowitz Model takes a narrow view, which is that it is premised on optimizing a single objective, it is undisputed that it is the most widely used model by researchers and practitioners in real world applications.

The focus on more than just a single-objective case of optimization (known as multi-objective optimization) is not only realistic, but also equitable, since it allows consideration of various criteria or expectations related to the problem to be considered in the optimization and thus is more representative for the many competing interests. Due to the obviously conflicting and incommensurable character of the criteria that may be involved, such problems have many good solutions, instead of just a unique optimal solution (Miettinen & Mäkelä, 2002), and solution algorithms have been designed to find them. Without loss of generality, the multi-objective problem is denoted by:

$$min\ f(x) = [f_1(x), f_2(x), \dots, f_k(x)]$$

$$\text{Subject to} \quad x \in X \subset R^n \qquad (1.1)$$

where f_i is the i^{th} objective function ($i = 1,2, \dots, k$), x is an n vector of decision alternatives and X is the set of feasible decision alternatives, also called feasible decision set, in which all the constraints are satisfied. The vector function $f(x)$ defines a criterion set in the space R^k from which points in the feasible decision set are mapped (Deb 2001; Miettinen & Makela, 2002). Unlike single objective problems, (1.1) as stated, has many solutions and therefore the notion of optimality as known in single objective optimization takes on entirely different meanings, and the most common and popular is Pareto optimality as defined (see Miettinen, 2000):

A solution $x^* \in X$ is said to be Pareto optimal (or non-dominated)
if $f(x^*) \leq f(x)$ for all $x \in X$ and there exists $x \in X$ such that $f(x^*) < f(x)$ (1.2)

The problem (1.1) may be solved in diverse ways. The choice of a method is dictated by the nature of the problem and the expectations of the analyst or decision maker. The methods are classified as either Scalar or Pareto (Deb, 2001). The scalar methods are suitable for continuous, differentiable and deterministic problems, while the Pareto methods are suitable for problems which depart from these basics, and therefore use heuristic or meta-heuristic algorithms in the search for solutions (Deb, 2001). The Weighted-Sum is a scalar method suitable for solving the models reported in this paper. For a more comprehensive review of the methods the reader may see Marler and Arora (2010). The weighted sum method optimizes a convex combination of the objective functions using weights w_i as defined by (1.3):

$$min\ f(x) = w_1 f_1(x) + w_2 f_2(x) + \dots + w_k f_k(x) \qquad (1.3)$$

$$\text{Subject to } x \in X \text{ and } w_1 + w_2 + \dots + w_k = 1, \quad w_i > 0 \quad \forall\ i$$

where w_i is the weight of the i^{th} objective function $i = 1,2,\ldots,k$. By varying w_i, it is possible to generate Pareto optimal solutions (Marler and Arora, 2010). Consequently, the entire set of Pareto optimal solution may be generated. The method is suitable for convex problems; it is inefficient with non-convex ones as well as problems with many objective functions (Marler and Arora, 2010). It requires little or no input from the user.

In the next section, the investment models and methodology are discussed. The subsequent section presents and discusses the results of running the models under sensitivity analysis and under a joint model. Finally, the paper is concluded with some recommendations.

2. The Investment Model

This section reproduces in substance, but briefly, the portfolio optimization problem and model used by Akanyare & Twum (2022) for the sake of completeness. The parameters of the model which can be subject to variations in their values under a sensitivity analysis are also discussed in this section. The model is:

$$\begin{aligned} &max\ E(X) = \sum_{i=1}^{n} E_i x_i \\ &min\ Z(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij}^2 x_i x_j \\ &subject\ to\ \sum_{i=1}^{n} x_i \le M \\ &\qquad L_i \le F_{ij}(X) \le U_i, \quad j \in N \\ &\qquad x_i \ge 0,\ i = 1,2,\ldots,n \end{aligned} \tag{2.1}$$

where E_i is the expected return from investment i and defined in terms of its rate of return r_{ik} in the time period k in the past, as $E_i = \frac{1}{p}\sum_{k=1}^{p} r_{ik}$ $(k = 1,2,\ldots,p)$; x_i $(i = 1,2,\ldots,n)$ is the amount of money to put in the investment i. The measure of variance of the overall expected value $E(X)$ is given by the function $Z = \frac{1}{p}[(r_{ik} - E_1)x_1 + (r_{2k} - E_2)x_2 + \cdots + (r_{nk} - E_n)x_n]^2$
$= \sum_{k=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}(r_{ik} - E_i)(r_{jk} - E_j)x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ij}^2 x_i x_j$ and $\delta_{ij}^2 = \sum_{k=1}^{p}(r_{ik} - E_i)(r_{jk} - E_j) = \frac{1}{p}\sum_{k=1}^{p} r_{ik} r_{jk} - \frac{1}{p^2}(\sum_{k=1}^{p} r_{ik})(\sum_{k=1}^{p} r_{jk})$; M is the maximum amount of money to be invested; $F_{ij}(X)$ is constraint j in relation to investment i and represents restriction on the amount to be invested in terms of minimum and maximum values, respectively given by L_i and U_i.

The main parameters of the model are r_{ik}, M, L_i and U_i. The model (2.1) has up to np number of the parameter r_{ik}. While practically the values of the parameters M, L_i and U_i are determined on the bases of a decision maker's preferences, experiences, or informed judgment (and may not be necessarily required to be varied by the decision maker, as the case is in this current work), the values of the parameter r_{ik} are estimates derived from historical data which is impacted by trends in the business and investment environment and therefore working with just a single fixed value for each of them is unreasonable. In this work, therefore, they are the parameters varied in the sensitivity analysis.

Since (2.1) is a convex optimization problem (see Akanyare & Twum, 2022), it can be solved efficiently by any scalar method, such as the Weighted-Sum Scalarization. Therefore, the weighted-sum method corresponding to the model (2.1) is given as in Akanyare & Twum, (2022) by:

$$\begin{aligned} & max\ [w_i E(X) - (1 - w_i)(Z(X))] \\ & \text{subject to } \sum_{i=1}^{n} x_i \leq M \\ & L_i \leq F_{ij}(X) \leq U_i, \quad j \in N \\ & x_i \geq 0, \ w_i > 0, \ \sum w_i = 1, \ i = 1,2, \dots . n \end{aligned} \tag{2.2}$$

As a generating or posteriori method (2.2) may be used to generate a set of the Pareto optimal solutions through repeated sets of weights variations, even without the involvement of a decision maker.

Objective functions may be normalized in multi-objective situations, so as to make them dimensionless and thus provide appropriate basis for comparison of their values in the solution search process. A notable transformation approach used in this work, is:

$$f_i^{norm} = \frac{f_i(x)}{|f_i{}^{opt}|}, \quad 0 \leq f_i^{norm} \leq 1, \ i = 1,2, \dots, k \tag{2.3}$$

where $f_i(x)$ and $|f_i{}^{opt}|$ are respectively the i^{th} objective function value and its unique positive optimum value. Their ratio yields a normalized objective function f_i^{norm}.

A technique used in Akanyare & Twum (2022) for assessing the Pareto optimal solutions to aid decision making in terms of identifying a preferred solution is a ranking scheme which provides a single measure for the Pareto optimal objective function values in each solution. Specifically, it is a Risk to Expected Return Profile (RERP) measure given by:

$$RERP = \left(\sqrt{Z(x)} \Big/ E(x) \right) * 100\% \tag{2.4}$$

which compares the standard deviation to the expected return for a Pareto optimal solution. It provides an objective basis for selecting a particular Pareto optimal solution for implementation, on account of the fact that lower values are better than higher values when comparing both values of the objective functions.

3. Results and Discussions

3.1 Introduction

In Akanyare & Twum (2022) the detailed description of the nature of the investment problems of the two businesses in Ghana and the data are given. In summary, the two businesses, referred to respectively as Investor A and Investor B, both engage in the purchase and sale of a variety of goods for profit. In the period studied, while Investor A had four areas of investments (denoted as A1, A2, A3, and A4) with projected rates of return over an eight months period, Investor B had three main areas (denoted as B1, B2, and B3) which are distinct from those of Investor A, with their projected rates of return over a twelve-month period. They both had fixed amounts to invest over their respective time periods and individual policies on their investments, and their desire was to determine how much to invest in the identifiable areas of investment in order to maximize return and minimize at the same time variability in the returns over their respective periods. For the sake of completeness, the separate models for the two investors are reproduced here as in Akanyare & Twum (2022).

Investor A: The Bi-criteria optimization model for investor A is:

$$max\ E_A(x) = 87x_{A1} + 92x_{A2} + 87x_{A3} + 85x_{A4}$$

$$min\ Z_A(x) = (x_{A1}\ x_{A2}\ x_{A3}\ x_{A4})\begin{bmatrix} 2.9 & 1.2 & -0.8 & 0.7 \\ 1.2 & 43.5 & 2.3 & -2.2 \\ -0.8 & 2.3 & 18.5 & -4.9 \\ 0.7 & -2.2 & -4.9 & 15.2 \end{bmatrix}\begin{pmatrix} x_{A1} \\ x_{A2} \\ x_{A3} \\ x_{A4} \end{pmatrix}$$

Subject to: $x_{A1} + x_{A2} + x_{A3} + x_{A4} \leq 250000$

$$x_{A1} + x_{A2} \leq 90000;$$
$$x_{A3} + x_{A4} \leq 90000;$$

$$x_{A1} \geq 5000;\ \ x_{A2} \geq 25000;\ \ x_{A3} \geq 50000;\ \ x_{A4} \geq 40000$$

The weighted Sum scalarized model for Investor A, involving normalized objective functions, is:

$$max[\ w_i E_A(x)^{norm} + (1 - w_i)(-Z_A(x)^{norm})]$$

Subject to: $x_{A1} + x_{A2} + x_{A3} + x_{A4} \leq 250000$

$$x_{A1} + x_{A2} \leq 90000;$$
$$x_{A3} + x_{A4} \leq 90000;$$

$$x_{A1} \geq 5000;\ \ x_{A2} \geq 25000;\ \ x_{A3} \geq 50000;\ \ x_{A4} \geq 40000$$

$$w_i + (1 - w_i) = 1, \quad w_i > 0\ \ \forall\ \ i$$

Investor B: The model for Investor B is:

$$max\ E_B(x) = 180x_{B1} + 203x_{B2} + 210x_{B3}$$
$$min\ \ Z_B(x) = 25x_{B1}^2 + 10x_{B1}x_{B2} + 4.2x_{B1}x_{B3} + 13.7x_{B2}^2 + 9.6x_{B2}x_{B3} + 17.9x_{B3}^2$$

Subject to: $x_{B1} + x_{B2} + x_{B3} \leq 220000$

$$x_{B1} + x_{B2} \leq 90000$$

$$x_{B1} + x_{B3} \leq 100000$$
$$x_{B2} + x_{B3} \leq 200000$$
$$30000 \leq x_{B1} \leq 50000$$
$$25000 \leq x_{B2} \leq 90000$$
$$20000 \leq x_{B3} \leq 200000$$

The weighted Sum scalarized model for Investor B with normalized objective functions, is similarly constructed as in the case of investor A.

3.2 Sensitivity Analysis

A primary parameter of the model likely to experience variation is the rate of return-on-investment parameter r_{ik}. A secondary parameter is the weight w_j associated with the weighted sum solution method which was found to have little or no impact at all on the model.as far as generating varied solutions was concerned (see Akanyare & Twum, 2022). Therefore, only the rate of return parameter is considered under the sensitivity analysis; the rest of the parameters are assumed fixed, in line with the positions of the two businesses on the subject of parameter variations.

The rate of return parameters which relate to price variations of the products purchased and sold by the two businesses are assumed therefore to be subject to variations from their original values up to $\pm 10\%$ over the investment period. Therefore, the values the parameters for Investors A and Investor B were respectively computed in steps of $\pm 5\%$ up to $\pm 10\%$. Furthermore, it is assumed that the variations in the values of the parameters is random and could thus be upwards or downwards in no specific predictable order over the period of the investment. Therefore, a sampling scheme is adopted in which the respective computed return parameter values are randomly sampled over the range of values of 0% to $\pm 10\%$. Two sets of such samples are presented in Tables 3.1 and 3.3 in connection with those for Investor A and subsequently for Investor B in Tables 3.4 and 3.6. The resulting data sets and their corresponding solutions obtained under the respective models for Investor A and Investor B are constituted into Scenarios A1 and A2 for Investor A and Scenarios B1 and B2 for Investor B, as given next.

Scenario A1

Table 3.1: First Set of Sampled Parameters values: Investor A

K	r_{1k}	r_{2k}	r_{3k}	r_{4k}	$r_{1k}r_{2k}$	$r_{1k}r_{3k}$	$r_{1k}r_{4k}$	$r_{2k}r_{3k}$	$r_{2k}r_{4k}$	$r_{3k}r_{4k}$	r_{1k}^2	r_{2k}^2	r_{3k}^2	r_{4k}^2
1	11.4	20	11	12	228	125.4	136.8	220	240	132	129.96	400	121	144
2	10	11	18	9	110.0	180	90	198	99	62	100	121	324	81
3	15	9	9	11	135	135	165	81	99	99	225	81	81	121
4	11	20	11	10.5	220	121	115.5	220	210	115.50	121	400	121	110.3
5	10.5	4.8	9.5	14.3	50.4	99.75	150.15	45	68.64	135.85	110.25	23.04	90.25	204.5
6	9	4.8	4.5	16.5	43.2	40.5	148.5	21	79.2	74.25	81	23.04	20.25	272.3
7	10.5	11	11	5.5	115.5	115.5	57.75	121	60.5	60.5	110.25	121	121	30.25
8	9	9	11	4.5	81	99	40.5	99	40.5	49.5	81	81	121	20.25
Total	86.4	89.6	85	83.3	963.1	926.2	904.2	1006.2	896.8	828.6	958.46	1250.1	999.5	983.5

In this scenario, two distinct Pareto optimal solution sets corresponding to the weight sets {0.1, 0,9} and {0.95, 0.05} emerged from the optimization as displayed in Table 3.2.

Table 3.2: Pareto Optimal Solution under Scenario A1

Decision variable	Pareto optimal values and Weight Sets	
	{0.1, 0,9}	{0.95, 0.05}
x_{A1}	5,000	65,000
x_{A2}	25,000	25,000
x_{A3}	50,000	50,000
x_{A4}	40,000	40,000
$E_A(x)$	11,695,000	15,438,000
$Z_A(x)$	64,930,000,000	728,175,000,000
$S.D = \sqrt{Z_A(x)}$	254,810	269,847
RERP (%)	2.179	1.748

Table 3.2 shows that variation of the values of the parameters produced no significantly different Pareto optimal solutions from the original model (see Akanyare & Twum, 2022). In other words, the levels of investments to be made remained the same as in the original model. The decision variable values as far as the two weight sets are concerned, were the same. The objective function values which are the expected value of the investments and their variance varied, however, slightly from the original ones, due to changed coefficients of the objective functions resulting from the variations.

Scenario A2

Table 3.3: Second Set of Sampled Parameters values: Investor A

K	r_{1k}	r_{2k}	r_{3k}	r_{4k}	$r_{1k}r_{2k}$	$r_{1k}r_{3k}$	$r_{1k}r_{4k}$	$r_{2k}r_{3k}$	$r_{2k}r_{4k}$	$r_{3k}r_{4k}$	r_{1k}^2	r_{2k}^2	r_{3k}^2	r_{4k}^2
1	21	9	9	15	189	189	315	81	135	135	441	81	81	225
2	11	20	18	10	220	198	110	360	200	180	121	400	324	100
3	10.5	9.5	10	10	99.8	105	105	95	95	100	110.3	90.3	100	100
4	22.1	10	9.5	10	221	209.9	221	95	100	95	488.4	100	90.3	100
5	5.8	5.3	10	15.8	30.7	58	91.6	53	83.7	158	33.64	28.1	100	249.6
6	5.6	5.3	15	15.8	29.6	84	88.5	79.5	83.7	237	31.36	28.1	225	249.6
7	10.2	9	9.5	5.3	92.2	97.3	54.3	85.5	47.7	50.4	104.9	81	90.3	28.09
8	12.9	11.4	12	4.8	147	154.8	61.92	136.8	54.72	57.6	166.4	129.9	144	23.04
Total	99	80	93	87	1029.4	1096	1047.3	985.8	800	1013	1496.9	938.4	1154.5	1075.4

In this scenario, three distinct sets of solutions emerged from the weighting of the objective functions. The weight sets producing the solutions are {0.1, 0.9}, {0.95, 0.05) and {0.99, 0.01}. The solutions corresponding to the weight sets are presented in Table 3.4.

Table 3.4: Pareto Optimal Solutions for Scenario A2

Decision variables	Pareto Optimal Values and Weight Sets		
	{0.1, 0.9}	{0.95, 0.05)	{0.99, 0.01}
x_{A1}	23,385	23,350	23,307
x_{A2}	25,000	25,000	25,000
x_{A3}	50,000	50,000	50,000
x_{A4}	40,000	40,000	40,000
$E_A(x)$	12,445,115	12,445,080	12,445,042
$Z_A(x)$	115,000,000,000	578,76,000,000	48,519,000,000
$S.D = \sqrt{Z_A(x)}$	339,120	240,570	220,270
RERP (%)	2.725	1.933	1.769

In the set of solutions of Table 3.4, the amounts to investment remained the same across the three sets of solution for the investment areas A2, A3, and A4, except the amount to invest in the area A1, which achieved a sharp rise of at least 18,307 above its corresponding value of 5000 achieved under Scenario A1. The objective functions values varied accordingly, for the same reason given under Scenario A1.

Scenario B1

Table 3.5: First Sampled Rates of Return Values: Investor B

k	r_{1k}	r_{2k}	r_{3k}	$r_{1k}\, r_{2k}$	$r_{1k}r_{3k}$	$r_{2k}r_{3k}$	r_{1k}^2	r_{2k}^2	r_{3k}^2
1	21	14.3	22	300.3	426	314.6	441	204.49	488
2	19	15	9	285	171	135	361	225	81
3	13.5	13.5	15.8	182.25	213.3	213.3	182.3	182.3	249.6
4	22	22.5	10	495	220	225	484	506.3	100
5	20	9.5	9.5	190	190	90.25	400	90.3	90.2
6	8	19.8	18	158.4	144	356.4	64	392	324
7	13	18	33	234	429	594	169	324	108.9
8	13	15	20	195	260	300.	169	225	400
9	9	13.5	14.3	121.5	128	193	81	182.3	204.5
10	3	22	10	66	30	220	9	484	100
11	11	15	18	165	198	270	121	225	324
12	22	21	27.5	462	605	557.5	484	441	756.2
Total	174.5	199.1	207.1	2854.4	3051	3489.1	2965.3	3481.5	4202.6

In Scenario B1, three weight sets produced three distinct solutions as presented as Table 3.6.

Table 3.6: Pareto Optimal Solution for Scenario B1

Variables	Pareto optimal values and weight sets		
	{0.9, 0.1}	{0.05, 0.95}	{0.01, 0.99}
x_{B1}	30,000	30,000	30,000
x_{B2}	25,000	29,355	60,000
x_{B3}	20,000	20,000	29,016
$E_B(x)$	14,365,000	15,231,645	23,196,312
$Z_B(x)$	63,610,000,000	69,263,800,380	139,005,885,300

S.D=$\sqrt{Z_B(x)}$	252,210	263,180	372,835
RERP (%)	1.756	1.728	1.607

The three distinct solutions produced slightly varied decision variable values especially for the area B2, with the investment area B3 varying slightly in the third set of solutions. The set of solutions are comparable with the original model solution (see Akanyare & Twum, 2022). As usual, the objective function values varied due to the variations in the coefficients of the objective functions resulting from the parameter variations.

Scenario B2

Table 3.7: Second Sampled Parameter values: Investor B

K	r_{1k}	r_{2k}	r_{3k}	$r_{1k}\, r_{2k}$	$r_{1k}r_{3k}$	$r_{2k}r_{3k}$	r_{1k}^2	r_{2k}^2	r_{3k}^2
1	21	15	19	315	399	285	441	225	361
2	20	14.3	9	286	180	128.7	400.	204.49	81
3	15.8	14.3	15	225.94	237	214.5	249.64	204.49	225
4	20	22.5	11	450	220	247.5	400	506.25	121
5	19	9	10	171	190	90	361	81	100
6	11	17	18	187	198	306	121	289	324
7	10.5	17.5	30	183.75	315	525.	110.25	306.25	900
8	14.3	18	14.3	257.4	204.49	257.4	204.49	324	204.49
9	9	15	15	135	135	225	81	225	225
10	4	13.5	9	54	36	121.	16	182.2	81
11	9	13.5	21	121.5	189	283.5	81	182.25	441
12	18	21	28.5	378	39	598.5	324	441	812
Total	171.6	190.6	199.8	2764.6	2816.5	2789.4	2789.4	3171	3875.7

This scenario produced the same Pareto optimal solutions for all the weight combinations applied. Therefore, a single solution corresponding to an arbitrarily selected weight set is presented in Table 3.8.

Table 3.8: Pareto Optimal Solution for Scenario B2

Variables	Optimal Pareto Values and a Weight set
x_{B1}	30,000
x_{B2}	60,000
x_{B3}	70,000
$E_B(x)$	30,570,000
$Z_B(x)$	22,600,000,000
S.D=$\sqrt{Z_B(x)}$	150,333
RERP (%)	0.492

3.3 A Joint Model for Investors A and B

An investigation is conducted in this section to assess whether or not it would be more profitable for the two businesses to operate as a single entity. To undertake this investigation, a composite rate of return is assumed and used to compute the required parameters for the joint model. The necessary terms are presented in Table 3.9.

Table 3.9: Parameter values for the Joint investment for Investors A and B

J \ k	1	2	3	4	5	6	7	8	9	10	11	12	Total
r_1	12	10	15	10	10	10	10	10	0	0	0	0	87
r_2	20	10	10	20	5	5	10	12	0	0	0	0	92
r_3	10	20	10	10	10	5	10	12	0	0	0	0	87
r_4	15	10	10	10	15	15	5	5	0	0	0	0	85
r_5	20	20	15	20	20	10	15	15	10	5	10	20	180
r_6	15	15	15	25	10	18	20	15	15	20	15	20	203
r_7	20	10	15	10	10	20	30	20	15	10	20	30	210
r_1r_2	240	100	150	200	50	50	100	120	0	0	0	0	1010
r_1r_3-	120	200	150	200	100	50	100	120	0	0	0	0	1040
r_1r_4	180	100	150	100	150	150	50	50	0	0	0	0	930
r_1r_5	240	200	225	200	200	200	200	150	0	0	0	0	1415
r_1r_6	180	150	225	200	100	180	200	180	0	0	0	0	1385
r_1r_7	240	100	225	100	100	200	300	200	0	0	0	0	1465
r_2r_3	200	200	100	100	150	25	100	144	0	0	0	0	1019
r_2r_4	300	100	100	100	75	75	50	60	0	0	0	0	860
r_2r_5	400	200	150	400	100	50	150	180	0	0	0	0	1630
r_2r_6	300	150	150	500	50	90	200	180	0	0	0	0	1620
r_2r_7	400	100	150	200	50	100	300	240	0	0	0	0	1540
r_3r_4	150	200	100	100	150	75	50	60	0	0	0	0	885
r_3r_5	200	400	150	200	200	50	150	180	0	0	0	0	1530
r_3r_6	150	300	150	250	100	90	200	180	0	0	0	0	1420
r_3r_7	200	200	150	100	100	100	300	240	0	0	0	0	1390
r_4r_5	300	200	150	200	300	150	75	75	0	0	0	0	1450
r_4r_6	225	150	150	250	150	270	100	75	0	0	0	0	1370
r_4r_7	300	100	150	100	150	300	150	100	0	0	0	0	1350
r_5r_6	300	300	225	500	300	180	300	225	150	100	150	400	3130
r_5r_7	400	200	225	200	200	200	450	300	150	50	200	600	3175
r_6r_7	300	150	225	250	100	360	600	300	225	200	300	600	3610
r_1^2	144	100	225	100	100	100	100	100	0	0	0	0	969
r_2^2	400	100	100	400	25	25	100	144	0	0	0	0	1294
r_3^2	100	400	100	100	100	25	100	144	0	0	0	0	1094
r_4^2	225	100	100	100	225	225	25	25	0	0	0	0	1025
r_5^2	400	400	225	400	400	100	225	225	100	25	100	400	3000
r_6^2	225	225	225	625	100	324	400	225	225	400	225	400	3599
r_7^2	400	100	225	100	100	400	900	400	225	100	400	900	4250

Table 3.9 is the result of combining the individual investments of the two investors into a single investment problem with seven (7) areas of investments denoted by $J1, J2, \ldots, J7,$ with their corresponding rates of return over a twelve (12) month period. Under the investment period, an investment area that had no rate of return for a given month was assigned a zero (0) rate of return.

The available amounts, or capital, of the two businesses are combined while maintaining their individual operational restrictions or policies as spelt out under their separate investments.

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Table 3.10 is constructed to account for the joint investment insofar as computing the expected values and covariance matrix for the objective functions are concerned. The associated expected value and variance expressions and composite constraints result in the joint model given by:

$$maxE_j(x) = 87x_{j1} + 92x_{j2} + 87x_{j3} + 85x_{j4} + 180x_{j5} + 203x_{j6} + 210x_{j7}$$

$$min\ Z_j(x) = 28x_{j1}^2 + 49x_{j2}^2 + 39x_{j3}^2 + 35x_{j4}^2 + 25x_{j5}^2 + 14x_{j6}^2 + 48x_{j7}^2 + 58x_{j1}x_{j2} + 68x_{j1}x_{j3} + 52x_{j1}x_{j4} + 18x_{j1}x_{j5} - 14x_{j1}x_{j6} - 10x_{j1}x_{j7} + 58x_{j2}x_{j3} + 34x_{j2}x_{j4} + 164x_{j2}x_{j5} + 10x_{j2}x_{j6} - 12x_{j2}x_{j7} + 44x_{j3}x_{j4} + 38x_{j3}x_{j5} - 8x_{j3}x_{j6} - 22x_{j3}x_{j7} + 30x_{j4}x_{j5} - 12x_{j4}x_{j6} - 24x_{j4}x_{j7} + 14x_{j5}x_{j6} + 4x_{j5}x_{j7} + 10x_{j6}x_{j7}$$

Subject to: $x_{j1} + x_{j2} + x_{j3} + x_{j4} + x_{j5} + x_{j6} + x_{j7} \leq 470000;$
$x_{j1} + x_{j2} \leq 100000;$
$x_{j1} + x_{j3} \leq 100000;$
$x_{j1} + x_{j7} \leq 100000;$
$x_{j2} + x_{j3} \leq 120000;$
$x_{j3} + x_{j4} \leq 100000;$
$5000 \leq x_{j1} \leq 60000;$
$25000 \leq x_{j2} \leq 50000;$
$5000 \leq x_{j3} \leq 80000;$
$40000 \leq x_{j4} \leq 60000;$
$30000 \leq x_{j5} \leq 50000;$
$25000 \leq x_{j6} \leq 50000;$
$20000 \leq x_{j7} \leq 60000$

3.4 Results of the Joint Model

In a similar manner as was done in the separate models, a scalarized form of the current model with normalized objective functions was solved to generate Pareto optimal solutions. In this case four weight sets (see first row of Table 3.10) yielded distinct solutions, which are presented in Table 3.10

Table 3.10: Pareto Optimal Solution for the Joint Model

Variables	{0.9, 0.1}	{0.8, 0.2}	{0.7, 0.3}	{0.05, 0.95}
x_{j1}	5,000	5,000	5,000	40,000
x_{j2}	25,000	25,000	25,000	25,000
x_{j3}	5,000	5,000	5,000	40,000
x_{j4}	40,000	40,000	40,000	60,000
x_{j5}	30,000	30,000	30,000	50,000

x_{j6}	25,000	25,000	25,000	50,000
x_{j7}	20,280	28,954	33,678	60,000
$E_j(x)$	22,353,800	23,125,340	24,173,800	46,110,000
$Z_j(x)$	306,480,000,000	312,198,000,000	321,430,000,000	114,960,000,000
S.D=$\sqrt{Z_j(x)}$	553,606	558,746	566,948	1,072,194
RERP (%)	2.476	2.416	2.345	2.325

Table 3.11 presents a further processing of the results of both the Separate and the Joint models, to facilitate discussion of the outcomes of the joint modeling. It shows all the thirteen Pareto optimal solutions in terms of the total amounts to invest, the total return on the investments, the profit, the overall risk (given by the standard deviation) of the return on the investments, and the Risk-to-Expected-Return-Profile. The bolded numbers identify the largest and least aggregate investment amounts for the joint model and the separate models, together with their corresponding overall returns, overall profits, overall risks, and RERP figures. For instance, the highest and least aggregate investment amounts for the joint model are 325,000 and 150,280 respectively. Those for Investor A are 180,000 and 120,000 respectively; while those for Investor B are 160,000 and 75,000 respectively.

Table 3.11: Processed Separate and Joint Models Results

	Solution	Investment	Return	Profit	Risk	RERP (%)
JOINT MODEL RESULTS	1st	**150,280**	**22,353,800**	**22,203,520**	**553,606**	**2.476**
	2nd	158,954	23,125,340	22,966,386	558,746	2.416
	3rd	163,678	24,173,800	24,010,122	566,948	2.345
	4th	**325,000**	**46,110,000**	**45,785,000**	**1,072,194**	**2.325**
SEPARATE MODELS RESULTS	Inv A 1st	**120,000**	**11,695,000**	**11,575,000**	**254,810**	**2.179**
	Inv A 2nd	**180,000**	**15,438,000**	**15,258,000**	**269,847**	**1.748**
	Inv A 3rd	138,385	12,445,115	12,306,730	339,120	2.725
	Inv A 4th	138,350	12,445,080	12,306,730	240,570	1.933
	Inv A 5th	138,307	12,445,042	12,306,735	220,270	1.769
	Inv B 1st	**75,000**	**14,365,000**	**14,290,000**	**252,210**	**1.756**
	Inv B 2nd	79,355	15,231,645	15,152,290	263,180	1.728
	Inv B 3rd	119,016	23,196,312	23,077,296	372,835	1.607
	Inv B 4th	**160,000**	**30,570,000**	**30,410,000**	**150,333**	**0.049**

3.4 Discussions

Sensitivity Analysis. The limited sensitivity analysis on the two separate models indicated that they were generally stable under slight parameter variations, which were confined to the rate of return parameters only. The fact that randomly selected combinations of the values of the varied rates of return parameters nevertheless resulted in slightly varied Pareto optimal solutions, is testimony to the observation made. Variation of the return rate parameters (within the levels of variations used) resulted almost in no variations in the Pareto optimal solutions in almost all the cases. However, the expected returns over the investment periods and the variance or standard deviation of the returns varied, sometimes quite largely (which is understandable), due to variations in their computed coefficients resulting from the parameter variations.

This observation means that the two businesses could expect to achieve comparable returns and profits with the levels of investments given by the solutions, even if the rate of return parameters actually varied within the range of $\pm$ 10%. In view of the stability of the separate models observed, the Joint model was assumed to be equally stable and therefore was exempted from the exercise.

Joint investment Model. The conceived joint investment problem produced results that indicate that the two businesses could make a higher combined profit on their joint investments than the sum of their separate profits. This can be seen from Table 3.11 where the highest investment amount for investing in the joint model is 325,000, whereas the highest amounts for the separate investments are 180,000 and 160,000, giving a sum of 340,000, which is higher than the joint investment amount by 15,000. On the other hand, the corresponding profit for the joint model is 45,785,000, whereas the corresponding combined profits of the separate models is 45,668,000, which is less than the profit of the joint model by 117,000. Therefore, with less than the sum of the separate investment amounts, the joint model can yield a profit higher than the combined separate profits of the two businesses. In terms of risk and RERP, however, the joint model presents higher values of 1,072,194 and 2.325 respectively against a combined separate risk of 420,180 and RERP of 1.797. This goes to show that while the joint investment may be more profitable than separate investments, it is riskier.

Another observation is that at the lowest level of investment, the joint model was much less profitable than the separate combined investments, and the risk higher for a joint investment than for even a combined separate investment. This can be seen using a similar argument as above for the lowest investment amount as was done for the highest investment amount. This observation may be indication that a joint investment is not profitable at the lowest level of investment, and poses even higher risk.

4. Conclusions

This work has been a further development to an earlier work by the authors, which took a multi-objective optimization approach to Portfolio optimization, instead of the traditional single objective approach, in the context of two businesses in Ghana. The optimization models of the two businesses were investigated in a post-optimality analysis on one hand and a joint modeling on the other. The two separate models were subjected to sensitivity analyses to assess how variations of the rate of return parameters affected the solutions under varied weights of the objective function. The outcomes revealed that the model was quite stable under parameter variations as virtually the same Pareto optimal solutions were obtained.

Furthermore, investigation into whether a joint venture would be profitable for the two investors was undertaken. The available amounts of the two investors were assumed to be used for the joint investment and a composite model developed that took into account their investment policies or constraints. The joint problem produced a maximum return higher than the combined returns of the separate models and a minimum risk higher than their combined separate risks. This is indication that on the basis of the conceived joint investment problem, it can be profitable for the two businesses to pool their resources and work as a unit. However, they would need to recognize the risk involved and find a practical way to reduce or manage. It appears that lower levels of investment may not be profitable under a joint investment.

Furthermore, since the solutions of the model reveal that under their operational policies, whether separately or jointly, the investors did not need to invest all their available funds, it is indication that mathematical modeling and optimization can provide investors and portfolio managers in general information as to whether or not they have to invest all their available funds, in spite of their operational policies. This can mean efficient application of their available capital for investment. One weakness with the weighted sum method of solution is that the solution may not be evenly distributed on the Pareto front. Several weights and weight combinations may have to be used to generate or locate just a few of the Pareto optimal solutions. Therefore, other solution methods can be investigated as to the diversity of solutions they may produce.

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