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Inquiring and learning with DGS Cui-Rods: a proposal for managing the complexity of how primary-school pupils mathematically structure odd-even numbers

Stavroula Patsiomitou
Greek Ministry of Education and Religious Affairs
PhD, University of Ioannina
MEd, National and Kapodistrian University of Athens, **Greece**
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Abstract

Inquiry approaches using coloured manipulatives are a fruitful field for the investigation of mathematical concepts, embedded in a re-conceptualized, research-based curriculum. Manipulatives are designed to mediate between a particular mathematical concept and the way pupils learn that concept. Many researchers highlight the advantages of computer manipulatives including DGS manipulatives for teaching and learning. Moreover, the effectiveness of the Cuisenaire–Gattegno approach in the teaching and learning of mathematics in primary schools has been the subject of many math-investigations, supporting positive outcomes. In terms of the present study, it is interesting to mention the introduction of DGS Cui-Rods that I created in the Geometer's Sketchpad dynamic geometry environment. The main focus of the current study is a fundamental pattern structure of our number system: odd and even numbers. The study will propose a multiple representation approach to aid pupils understanding odd and even numbers. The proposed DGS material can be displayed, inquired and managed through properly set-up tasks, using linking representations. Finally, it is important to continue teaching concepts through activities, tasks and problems that involve children in the inquiring and learning process; it is the best route for them to how to develop, interpret, and make sense of mathematical concepts.

Keywords: Computer manipulatives, Cuisenaire-Gattegno rods, DGS Cui-Rods, even and odd numbers, number zero.

1. Introduction: The construction of knowledge through inquiry approaches

"*All depends upon the activity which the mind itself undergoes in responding to what is presented from without" (Dewey, 1902/1990, p. 209 cited in Jaworski, 2003)*

Pupils face difficulties when they explore mathematical objects, no matter if they are in a static or dynamic environment. They have to mentally operate on the abstract object, even if it is visually supported by a computing environment. This is what Laborde (2003) investigates, interrogates or asks: "but if the thought experiments on abstract objects are not available (as it is often the case for learners), a crucial question about learning is whether such environments could favour *an internalization process of the external actions in the environment"*. In a constructivist approach the reference to schemes is essential. Littlefield-Cook, & Cook (2005) write that

> *"For Piaget, the essential building block for cognition is the scheme. A scheme is an organized pattern of action or thought. It is a broad concept and can refer to organized patterns of physical action (such as an infant reaching to grasp an object), or mental action (such as a high school student thinking about how to solve an algebra problem). As children interact with the environment, individual schemes become modified, combined, and reorganized to form more complex cognitive structures" (p.6, in Chapter 5).*

Piaget (1937/1971) supports that pupils construct new concepts, 'assimilating' in a conservative way or 'accommodating' in a modifying way their prior knowledge conceptions. In the last chapter of his work *"The Construction of Reality in the Child"* Piaget (1937/1971) stated that: "[...] *Assimilation and accommodation are therefore the two poles of an interaction [...] and such an interaction presupposes from the point of departure an equilibrium between the two tendencies of opposite poles." (Pp.2-3)*

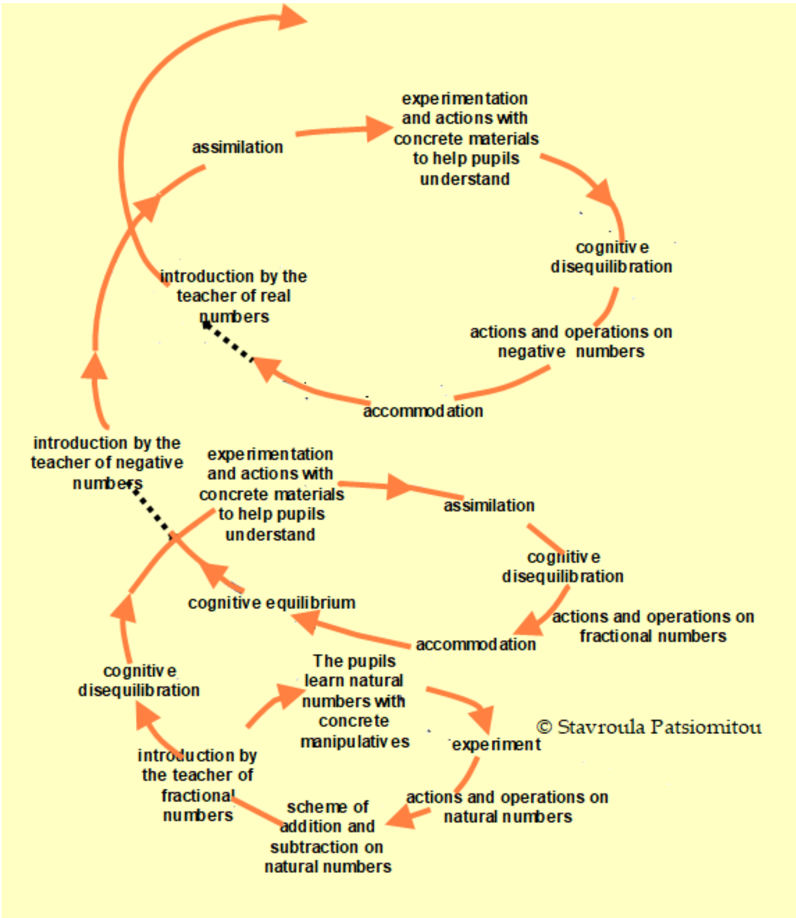

Fig. 1: My proposal for a spiral curriculum for the learning of numbers, taking into account the aforementioned notions of Piaget and Bruner (Patsiomitou, 2019a, p. 111)

Bruner (1966) developed an instructional theory. Bruner emphasized the teacher's proper use of language when they introduce a meaning to children. Discovery learning was also advocated by Bruner (1961, 1966). He pointed out that discovery learning "increases the interest of students, creates exciting classroom atmosphere, encourages and increases participation, provokes enthusiasm and inquiry, and helps students learn new content" (Bayram, 2004, p.40). Within the theory developed by Bruner (1966) cognitive conflict *"occurs when there is a mismatch between information encoded in two of the representational systems,* [...] what one sees and how one says it [...]" (Bruner, Olver, & Greenfield, 1966, p. 11). According to El Rouadi & Al Husni (2014, p. 130) *"Bruner focused on the spiral curriculum which can be explained as follows: learners acquire the basic ideas initially by using their intuition; and after words, the learner builds on them by revisiting these basic ideas as frequent as required until the meaningful understanding is fully achieved".* Figure 1 may be considered as a spiral curriculum for the learning of numbers, taking into account the aforementioned notions of Piaget and Bruner; how the learning of numbers occurs during the school years from primary to secondary and tertiary education (see also Patsiomitou, 2019a, p.111).

Why do young learners face difficulties when they explore mathematical objects? Let us look at the way pupils understand numbers. How do they construct the scheme of the *"sum of two numbers"?* In my opinion this process moves like a spiral, starting in the first years of a child's life and continually reiterating the process of assimilation and accommodation for every new concept that is learnt at increasingly abstract levels. The class in the first year of secondary education when teachers are obliged to introduce negative numbers to pupils is one of the more "difficult" parts of their teaching lives. This is because *"there is an imbalance between the new experience and the old scheme."* (Littlefield-Cook, & Cook, 2005, p.8, in Chapter 5). For this, Figure 1 may be thought of as a *spiral of equilibration*, trying to illustrate how pupils understand and integrate the ways to subtract numbers in several different phases of their learning life.

The young learners learn how to represent numbers using coloured manipulatives (e.g., Cuisenaire rods, abacuses, Dienes cubes, Montessori colour beads, fraction circles, Geoboards, pattern blocks). *"Mathematics educators have used them, as pedagogically oriented objects […] to immerse mathematical ideas in the feeling of their materiality"* (Nemirovsky, & Sinclair, 2020, p.107). These materials are an excellent tool which helps pupils overcome their cognitive obstacles. Since tools exert an influence over the technical and social way in which pupils conduct an activity, they are considered essential to their cognitive development. According to Vygotsky (1978), tools can be considered as external signs and they can become tools of semiotic mediation. Vygotsky developed the zone of proximal development (ZPD) and defined it as *"The distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers"* (p.86).

In Vygotsky's theory, it is taken for granted that less advanced pupils can learn from their peers who have more competence to solve problems and can interpret a meaning between representational systems. As I have written previously (e.g., Patsiomitou, 2008, 2019a, b), students construct *mental linking representations* as they interact with dynamic *linking visual* *active representations* (LVARs) in many different ways which are dependent on the individual student's conceptual understanding, and how well-developed their thinking competences and processes are. Goldin & Kaput (1996) depict an interaction between mental representations ("as those […] that are encoded in the human brain and nervous system and are to be inferred from observation") (p. 402) and external representations (e.g., written words, speech, formulas, concrete manipulatives, computer microworlds). Another possible point of view could have been to refer to conceptual metaphors. Conceptual metaphor theory was originally developed by Lakoff & Johnson (1980, 1999). A geometrical diagram can also be considered as a metaphor for the corresponding algebraic number. Sfard (1994) draws on the work of Lakoff & Johnson, reporting that a metaphor is "a mental construction which plays a constitutive role, in structuring our experience and in shaping our imagination and reasoning" (p. 46). A very useful method for helping pupils understand mathematical concepts is the use of the development of inquiring into practices of teaching. According to Jaworski (2003) "inquiry approaches "usually imply less formality than is expected of research" (p.2). "The inquiry approaches are no panacea for developing effective learning at any level […but] they have been shown as powerful to promote the kinds of thinking that lead to development" (Jaworski, 2003, p.7).

2. Visual representations as constructivist frame for the knowledge construction

Mathematics visualization and connections, links and relations between representations have appeared in literature as fundamental aspects to understanding pupils' construction of mathematical concepts, as well as important characteristics of learning and problem solving (e.g., Janvier, 1987a, b, c; Kaput, 1989). The constructivist view of representation as conceptual knowledge is consistent with the notion that learners actively construct new knowledge in problem solving situations *"when their current knowledge results in obstacles, contradictions, or surprises"* (Cobb, 1988, p. 92). The subject of this study is also linked with the notion of semiotic register developed by Duval (1996, 1999). The semiotic registers used in the mathematical activities are the algebraic, the graphical, the figurative and the natural language. A semiotic register, according to Duval, *constitutes a system of representation* if it allows three cognitive fundamental characteristics: its production, a treatment, and a conversion between different semiotic registers. Thus, the operative connections we expect during learning differ in their registers of semiotic representation. Goldin (1998) denotes the notion of "Representational systems" or "representational modes," as those systems *"which include systems of spoken symbols, written symbols, static figural models or pictures, manipulative models, and real-world situations, discussed by Lesh (1981)* […]" (p.143). Lesh, Post, & Behr (1987) proposed a multiple representation model in which they suggest a student understands a concept if s/he has the competence to translate between different modes of representation of the concept. (Figure 2). Behr, Lesh, Post, & Silver (1983) have identified five distinct types of representation systems that occur in mathematics learning and problem solving: (a) *"experience-based "scripts"*-in which knowledge is organized around "real world" events that serve as general contexts for interpreting and solving other kinds of problem situations; (b) *manipulatable models*-like […] arithmetic blocks, fraction bars, number lines, etc., in which the "elements" in the system have little meaning per se, but the "built in" relationships and operations fit many everyday situations; (c) *pictures or diagrams*-static figural models that,

[…] can be internalized as "images"; (d) *spoken languages*-including specialized sub languages related to domains like logic, etc.; (e) *written symbols*-which, like spoken languages, can involve specialized sentences and phrases (X+3=7, AUB) as well as normal English sentences and phrases' (reported in Lesh, Post & Behr, 1987) (Website [16]).

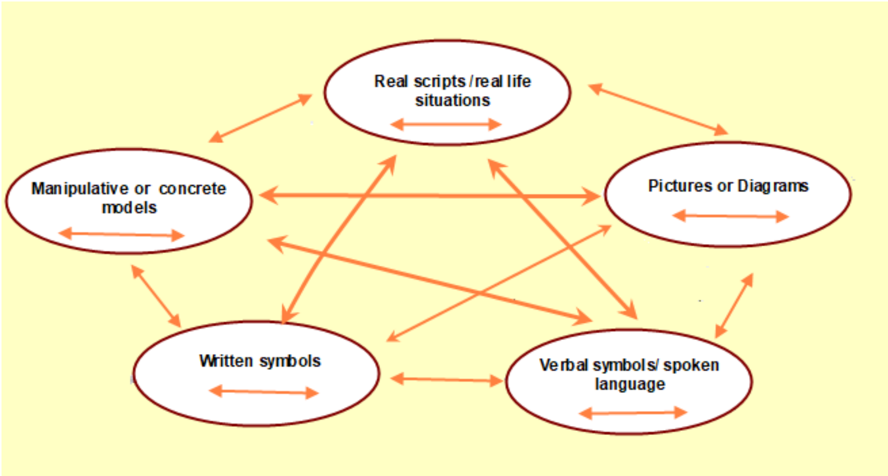

Fig. 2: Lesh's model (1979) adapted from Lesh, Post, and Behr (1987, p.34) (an adaptation for the current study).

Many similar figures have been constructed. For example, Lesh & Doerr (2003) replaced the "Real scripts/or Real-life situations" mode of representation with the "Experienced-based Metaphors", adding by this new information in the multiple representation figure. Post (1988) in his study *"Some notes on the nature of mathematics learning"* examines the implications that behavioral and cognitive theories have for the teacher in the mathematics classroom, as *"two broad theoretical umbrellas under which the vast majority of learning theories can be classified"* (p.1).

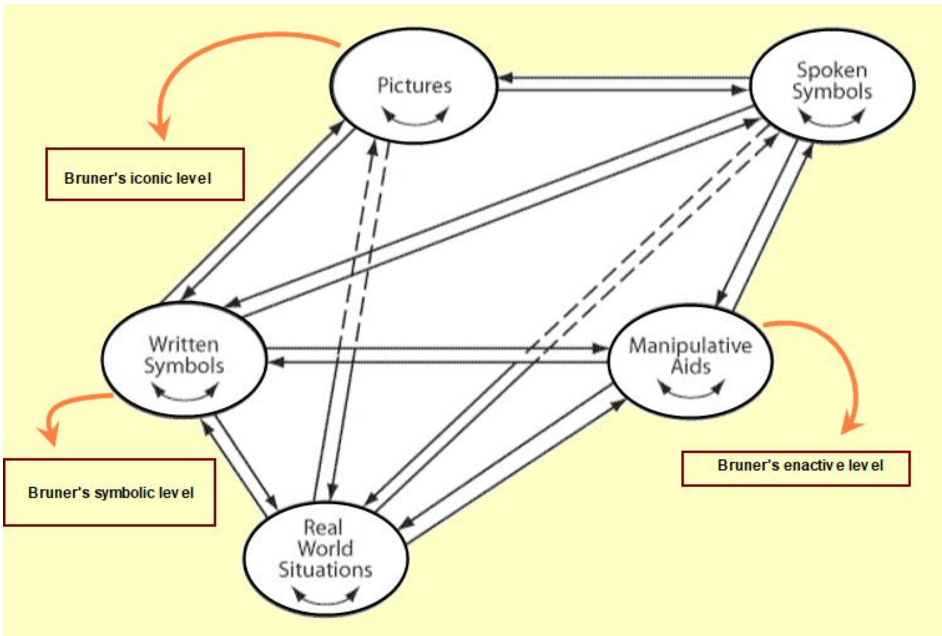

Fig. 3: Lesh's model (1979) (cited in Post, 1988, p.11) (an adaptation for the current study)

According to Post (1988): "When learning a new concept, it is important that pupils "see" the concept from a variety of perspectives or interpretations. […] These modes, [shown in Figure 3] represent an extension of Bruner's early work in representational modes (Bruner, 1966). The term *"manipulative aids"* in this figure relates to Bruner's enactive level, "pictures" relates to Bruner's iconic level, and "written symbols" relates to Bruner's symbolic level. […] Manipulative aids are in a sense halfway between the concrete real world of problem situations and the world of abstract ideas and mathematical symbols (written or oral). They are symbols in that they are made of physical materials, which in turn represent real-world situations" (p. 13).

We can represent a concept with multiple representations, such as pictorial representations, verbal representations, real-world representations, manipulatives or concrete representations, and symbolic representations (e.g., Vergnaud, 1988; Ainsworth, 1999a, b).

Verbal representations: These are representations that are generated through the language and verbal expressions we use while discoursing in a mathematics class.

Symbolic representations: These are representations which include/incorporate symbols such as letters, numbers, other symbols, formulas, operations on numbers and formulas, and arithmetic, algebraic or geometric symbols.

Real-world representations: These representations are correlated with situations, events and objects that take place in the real world.

A strong argument that a student cannot understand a concept from one type of representation of the concept alone is that this type of representation cannot describe a mathematical concept thoroughly-- each representation has its own distinct advantages. The core of mathematical understanding can thus be reached /achieved through the use of multiple representations. Kaput, Noss, & Hoyles (2002) also analysed *new representational infrastructures, namely "the ways we use to present and re-present our thoughts to ourselves and to others, (in order) to create and communicate records across space and time, and to support reasoning and computation".* The interaction with visual mathematical representations in a computing environment has two aspects: the learner acting upon it, and the visual mathematical representation responding or reacting in some form for the learner to interpret (Sedig & Sumner, 2006, p.5).

3. Physical and computer manipulatives

Manipulatives or concrete representations are objects (e.g., Cuisenaire rods) which are designed to mediate between a particular mathematical concept and the way pupils learn the concept. Pupils can manipulate them by touching or moving, and thus are concrete means (Dienes, 1960; Baroody, 1989; Van de Walle et al., 2005). Ross (2004) defines manipulatives as: "[…] *materials that represent explicitly and concretely mathematical ideas that are abstract. They have visual and tactile appeal and can be manipulated by pupils through hands-on experiences"* (p. 5). Generally speaking, a virtual manipulative is defined as *"an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge"* (Moyer et al., 2002, p. 373). Some researchers used dynamic geometry environments to investigate the understanding of mathematical concepts of primary school pupils (e.g., Ng & Sinclair, 2015, Sinclair & Crespo, 2006). Sarama & Clements (2016) discussed research on both, physical and virtual manipulatives *"to provide a framework for understanding, creating, implementing, and evaluating efficacious manipulatives"* (p.71). Nemirovsky & Sinclair (2020) also pose the following questions which are the focus of the special issue *"Digital Experiences in Mathematics Education":*

> *"How do the learning affordances of digital and tangible tools differ from each other? In what cases they are or aren't mutually substitutable? Are there optimal combinations of digital and tangible tools? How do tangible and digital tools entangle differently with the aesthetic and affective dimensions of mathematics learning? Are there sequences for their alternate use that appear to enhance learning experiences? What theoretical frameworks can help us understand their differences and complementarities?"* (p.108)

Numerous researchers present the advantages/ key benefits of using computer manipulatives, and rethink the meaning of "concrete" manipulatives. Clements & Mcmillen (1996), in their extended and substantial study *"Rethinking "concrete" manipulatives"* highlight the advantages of computer manipulatives for teaching and learning: *"Computers change the very nature of the manipulatives" (p.272)*. They argue that *"attitudes towards mathematics are improved when pupils are instructed with concrete materials by teachers knowledgeable about their use* […]" (p.270). They present the advantages/ key benefits of using computer manipulatives, and rethink the meaning of "concrete" manipulatives. Furthermore, in Clements & Mcmillen's (1996) opinion *"Computer manipulatives link the specific to the general, encourage problem posing and conjecturing, build scaffolding for problem solving, focus attention and increase motivation and encourage and facilitate complete, precise explanations"* (Clements & Mcmillen, 1996, p. 275-276).

Building Blocks also have been shown to considerably and substantially increase students' mathematical knowledge (Clements & Sarama, 2007). As I was investigating for computer manipulatives in Google, I found a plethora which I could use with young learners. Below I present a screenshot of these manipulatives as well as the website that allows visitors to find and experiment with them (Figure 4,5,6,7,8,9, 10).

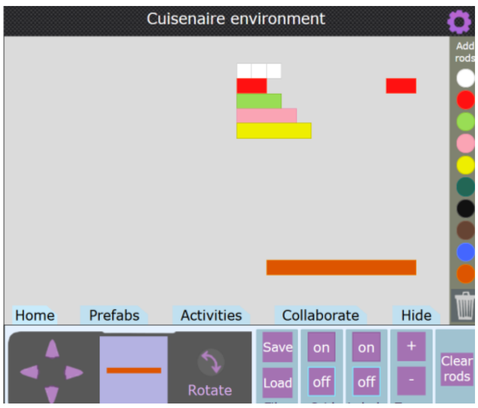

Fig. 4: *Webpage screenshot (website [1])*

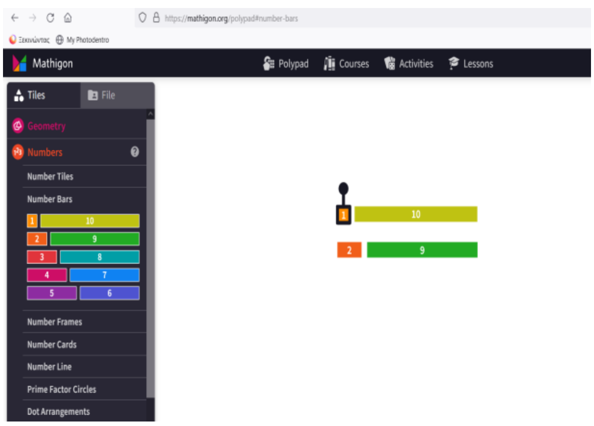

Fig. 5: *Webpage screenshot (website [2])*

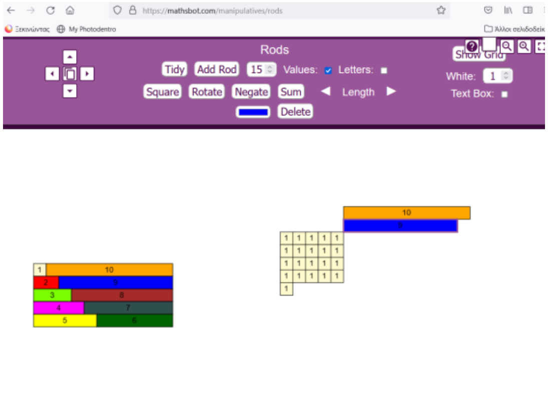

Fig. 6: *Webpage screenshot (website [3])*

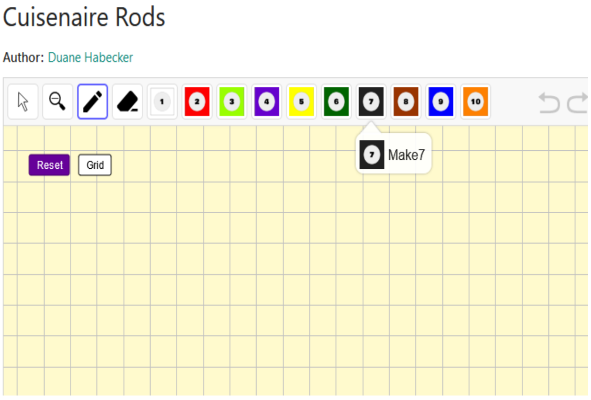

Fig. 7: *Webpage screenshot (website [4])*

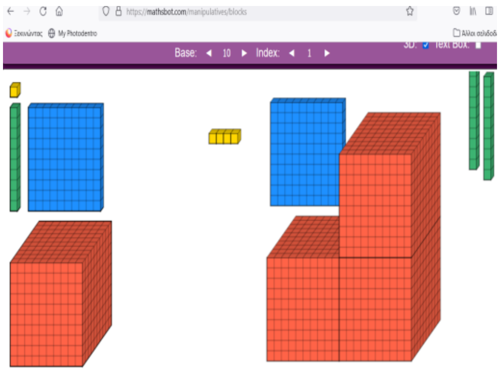

Fig. 8: *Webpage screenshot (website [5])*

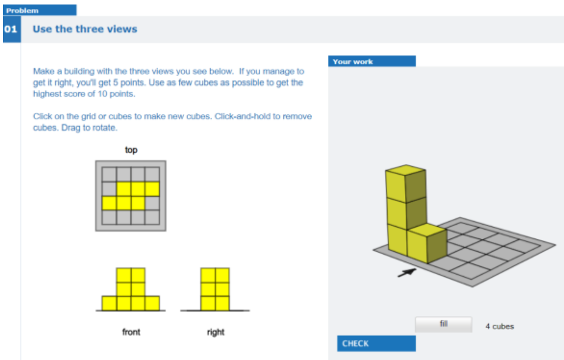

Fig. 9: *Webpage screenshot* (website [6]) (Freudenthal Institute)

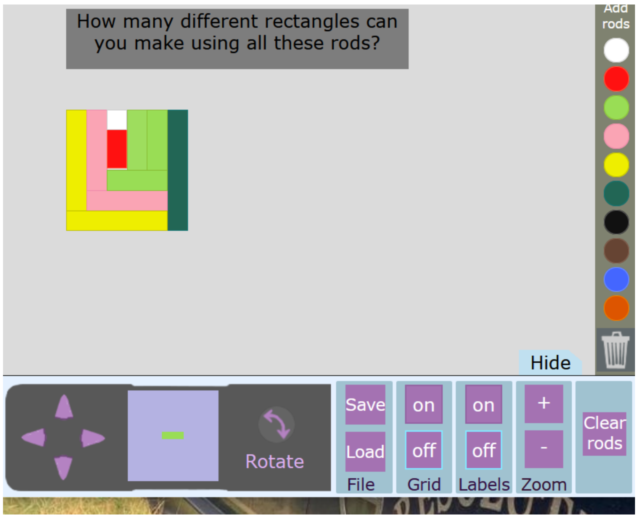

Fig. 10: *Webpage screenshot* (website [19])

Websites such as these mentioned above (e.g., websites [1]-[9]) are free and easy to access. They focus on a variety of mathematical areas such as number operations or geometry, using virtual manipulatives. Furthermore, these sites provide instructions how to use the virtual material (e.g., website [10], [17], [19]). Cuisenaire-Rods are mathematics learning aids for pupils. They provide an interactive, hands-on way to explore mathematics and learn mathematical concepts (e.g., the four basic arithmetical operations, working with fractions and finding divisors). Cuisenaire rods were invented in 1945 by a Belgian primary school teacher Georges Cuisenaire and popularised by Caleb Gattegno (Gattegno, 2011a, b). According to Benson et al. (2022) “Gattegno was a working mathematician and educator, and an early collaborator on mathematics teaching reform with the influential developmental psychologist Jean Piaget […who] identified human thought itself with logico-mathematical structures and held a rigorous view on how children would grow their understandings” (Benson et al., 2022, p. 3). According to Benson et al. (2022)

> *“The Cuisenaire–Gattegno approach to early mathematics uses colour coded rods of unit increment lengths embedded in a systematic curriculum designed to guide learners as young as age five from exploration of integers and ratio through to formal algebraic writing. The effectiveness of this approach has been the subject of hundreds of investigations supporting positive results, yet with substantial variability in the nature of results across studies”.* (p. 1)

In this study, I am presenting a DGS version of Cuisenaire rods (DGS Cui-Rods) that I created in the Geometer’s Sketchpad (Jackiw, 1991/2001) DGS environment. In the following section, we shall examine DGS Cui-rods in terms of the way in which pupils construct knowledge, as they are working on diagrams (semi) predesigned by the teacher and activated by the pupils.

4. The DGS Cui-Rods: inquiring even and odd numbers

As we build models of children's mathematical tasks and activities, “it is important to identify the used cognitive operations” (Wheatley, & Reynolds, 1996, p. 67). In terms of the current study, special attention was given to the construction of abstract units from rectangular shapes which represent the DGS Cui-Rods. Particularly, I created the Cui-Rods using parameters of concrete length (Figure 11). Below, I am describing the DGS Cui-rods construction in the Geometer’s Sketchpad dynamic geometry software and explain a few rules they follow (see also Patsiomitou, 2022, p.4):

- The rectangular shapes are constructed on the screen using the parametrical mode (i.e., using different parameters for each number). The Cui-Rod “1” is a square with sides equal to 1cm, created by the parameter “ONE= 1cm”. The Cui-Rod “2” is a rectangle with sides equal to 1cm and 2cm. So, both parameters, parameter “ONE” and parameter “TWO”, are used for its construction. This action gives the students the opportunity to visualize the difference in length as well as the value of the Cui-Rod. So, they can order the numbers 1<2<3.... etc.
- Every Cui-Rod can be moved from the "blue" point and can turn its orientation.
- The even numbers have been highlighted in yellow and the odd numbers in red, allowing the student to see the colours at once and visualize a categorization of the numbers in two discrete sequences.
- The students can form different numbers by adding rods. For this, the students can understand that the sum of 8+2 =10, 7+2 =9 etc. They can also construct different ways of creating a number. For example, 9=7+2=6+3 etc. The important thing here is that

they can visualize the numbers or the units on the rods as they investigate the ways to add or subtract numbers.

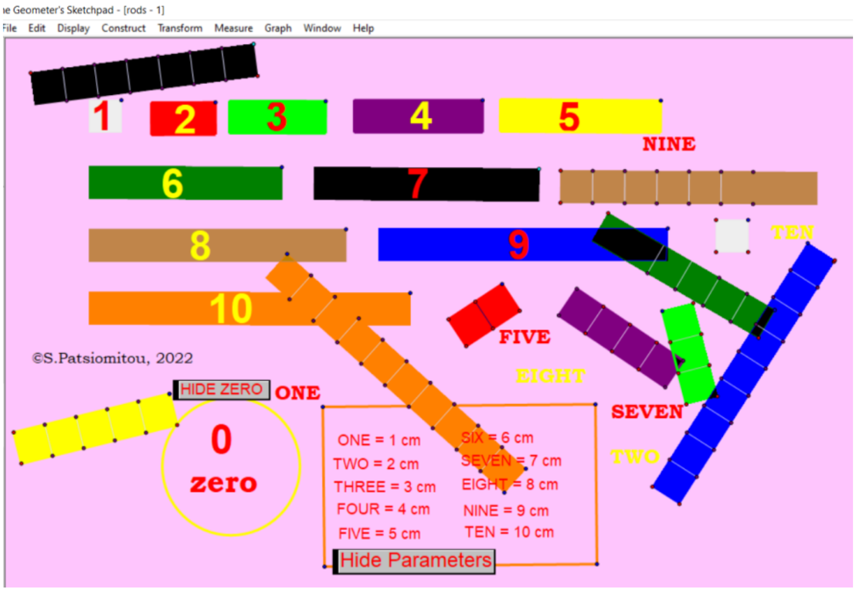

Fig. 11: The DGS Cui-Rods (Patsiomitou, 2022)

In order to comprehend the advantages (and disadvantages) of the construction mode in the dynamic geometry software, it is necessary to examine the differences between it and the mode of construction using static means. This will allow us to compare the two modes. For instance, in using a straightedge with measurements, the mode of constructing a figure in the software (e.g., a square of side equal to 1cm) could be different from the mode we use to construct it on paper. Duval (1999) argues that *"measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization."* (p.21). One such way would be to define '1' on the screen by using a new parameter and then use it as a radius of a circle in the construction. In this way, the sides of the square cannot be modified from the vertices of the shape using the dragging modality. Instead, they depend on the modification of the initially defined parameter (Patsiomitou, 2019b). The arbitrary segment '1cm' could thus be defined as a non-collapsible compass. As a consequence, the use of the DGS Cui-Rods depends on the teachers' geometrical knowledge of the relationships between the properties of figures. Moreover, this construction method induces a different mental perception for the teacher who creates/or uses the activity for the young learners. Many researchers suggest the importance of visuospatial processing, as well as the development of mental imagery in meaningful mathematical activities (e.g., Kosslyn, 1983; Wheatley, 1990). Young learners face difficulties to solve mental rotation tasks. Mental rotation is a key tool necessary to understand mathematical meanings. Piaget, & Inhelder (1967) argued that young learners are able to generate mental images of moving objects when they reach the stage of performing concrete actions (Shriki et al. 2017, p. 544).

On the next page, the corresponding parameter with the verbal expression of the number has been placed to the right (or the left) of each number (Figure 12). As we know whole numbers can be classified into even and odd numbers. Even numbers are those numbers that are divisible by 2. Dragging and placing the DGS Cui-Rods in pairs to make a bigger number is one of the games that can help pupils understand addition, but also the different ways they can get the same number.

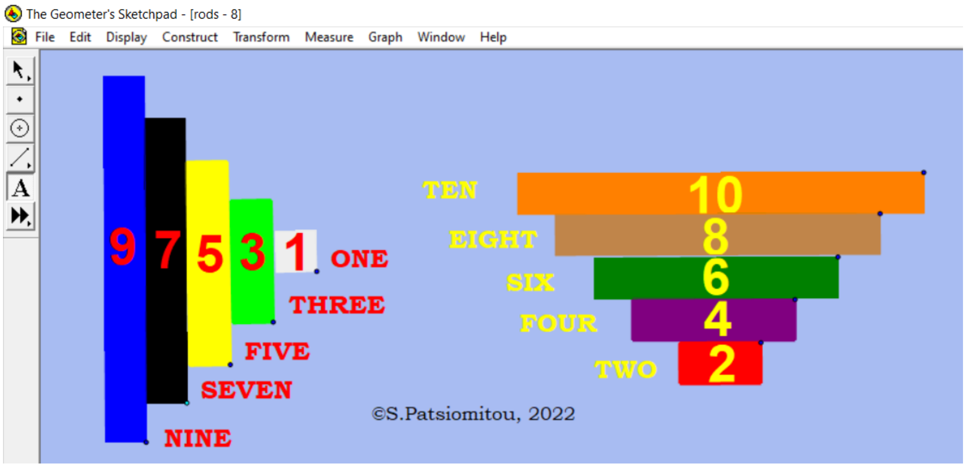

Fig. 12: Staircases with the Cui-Rods and mental rotation tasks

As we can see in the Figure 12, the Cui-Rods have a different screen layout: the even rods are placed on the left and the odd rods on the right. We are able or the DGS construction allows us to move the Cui-Rods into a vertical or horizontal position. With this placement, we make it possible for the student to notice that two units are missing each time to complete the next number. (3+2 =5, 7+2=9). Young learners begin counting using one-to-one correspondence: corresponding one object or number to another object or number [which belong to a set of the same elements]. According to Nunes & Bryant (2007)

> *"[…] if we are to pursue the approach of studying the links between children's quantitative reasoning and how they learn about natural numbers, we need to find out how well children understand the principle that sets which are in one-to-one correspondence with each other are equal in quantity"* (Nunes & Bryant, 2007, p. 8).

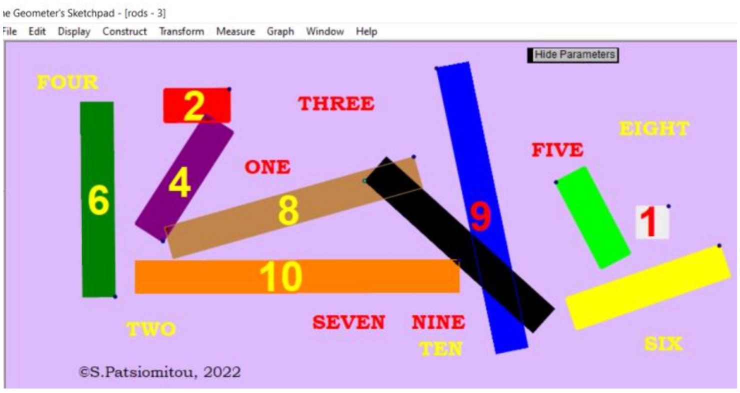

Fig. 13a: Even and odd numbers

In my opinion, pupils understand numbers when they realize that a quantity and its numerical representation are a one-to-one correspondence. Steffe et al. (1983) define units in the context of numbers as "a collection of individuals taken together […] a unitary item composed of a plurality of parts" (p. 6). Furthermore, *unitization* is the "cognitive *assignment* of a *unit* of *measure to a given quantity"* (Lamon, 1999, p.42).

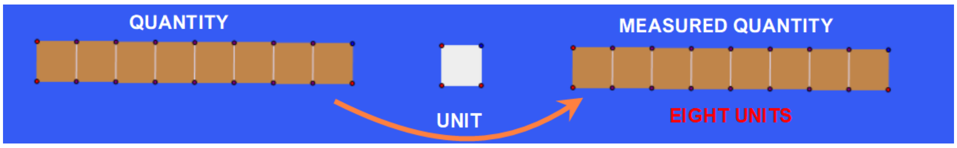

Fig. 13b: Measuring a quantity

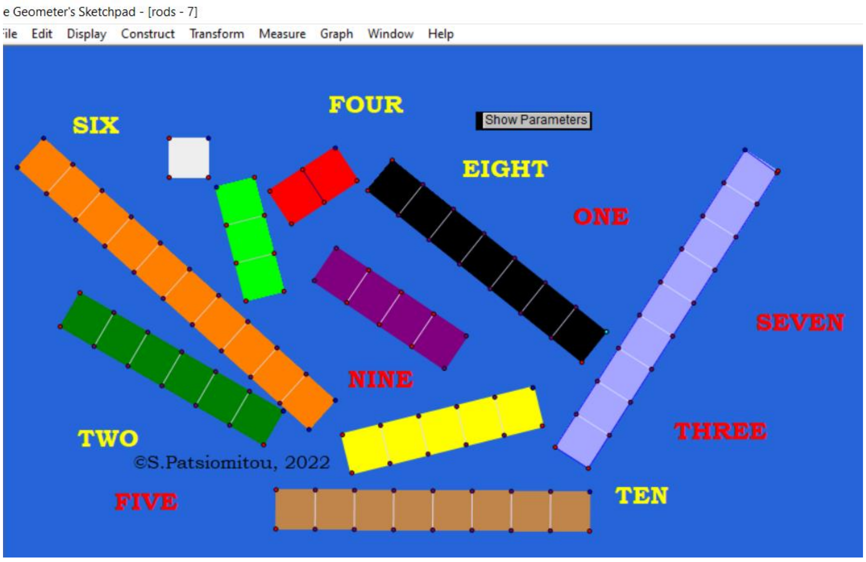

Fig. 13c: Equivalence and connection of the DGS Cui-Rods set in one-to-one correspondence to number's verbal expression

Placing the verbal expression of the numbers next to the corresponding symbol makes it possible for the pupil to connect the symbolic form of the number with its schematic and verbal or spoken expression. The "Find the number game" is just one of many exciting and interactive math games we can play when using DGS Cui-Rods (Figures 13a, b, c). As we can see there is a one-to-one correspondence between number labels and DGS Cui-Rods. The game challenges students to assign numerical values to each one of the Cui-Rods (e.g., corresponding the shape of a Cui-Rod with a symbolic number or its verbal expression).

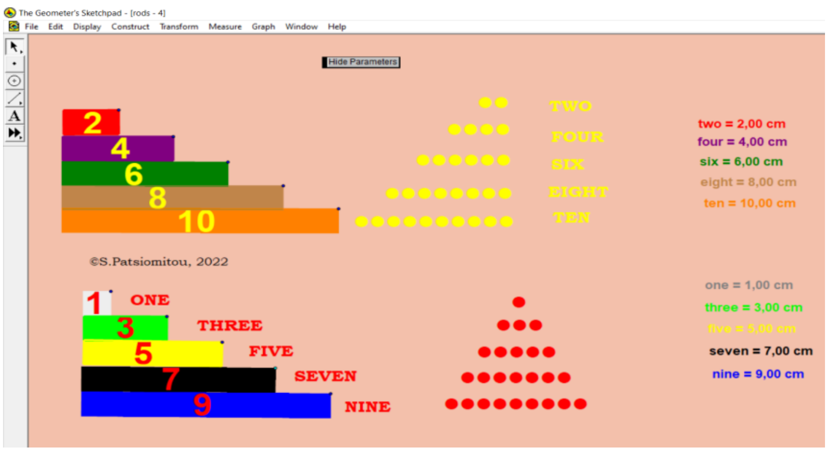

Fig. 14: Linking representations of the same number

Adding an equal number of dots next to each Cui-Rod results in the formation of multiple representations for the same number (Figure 14). Thus 2 is depicted schematically, with the number 2, but also verbally with "two", as by two dots.

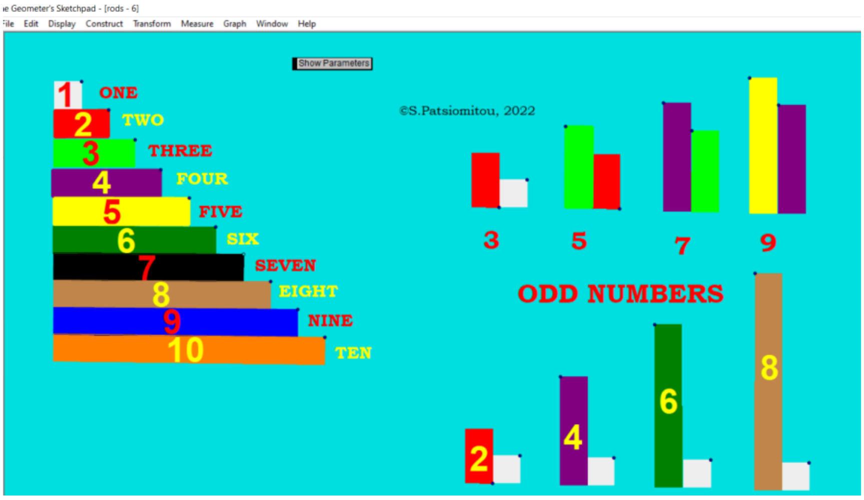

Fig. 15: Representing odd numbers

On the next page, all the numbers are placed on the staircase on the left (Figure 15). On the right, the odd numbers have been formed in two ways: (a) the previous even on the left, and the one on the right and (b) as the sum of an odd and an even (e.g., 3=2+1, 5= 3+2, 7= 4+3...). Adding an odd number and an even number results in an odd number. For example, in the screenshot (Figure 16) we have the addition of rods 1, 2 to equal the rod of length 3. The addition confirms the result and this is also captured as an operation between the triangles in which I have configured the colours to facilitate mental connections.

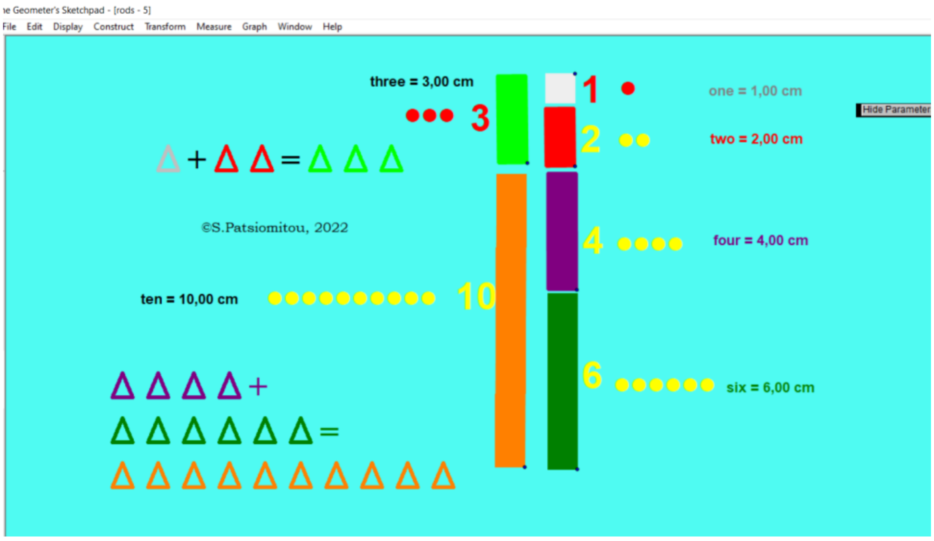

Fig. 16: Inquiring a few properties of the odd and even numbers

According to Nunes & Bryant (2007) “Adding and subtracting elements to sets also give children the opportunity to understand the inverse relation between addition and subtraction” (p. 4). On the next page (Figure 17) pupils are able to inquire a multiple representation approach, using the DGS Cui-Rods.

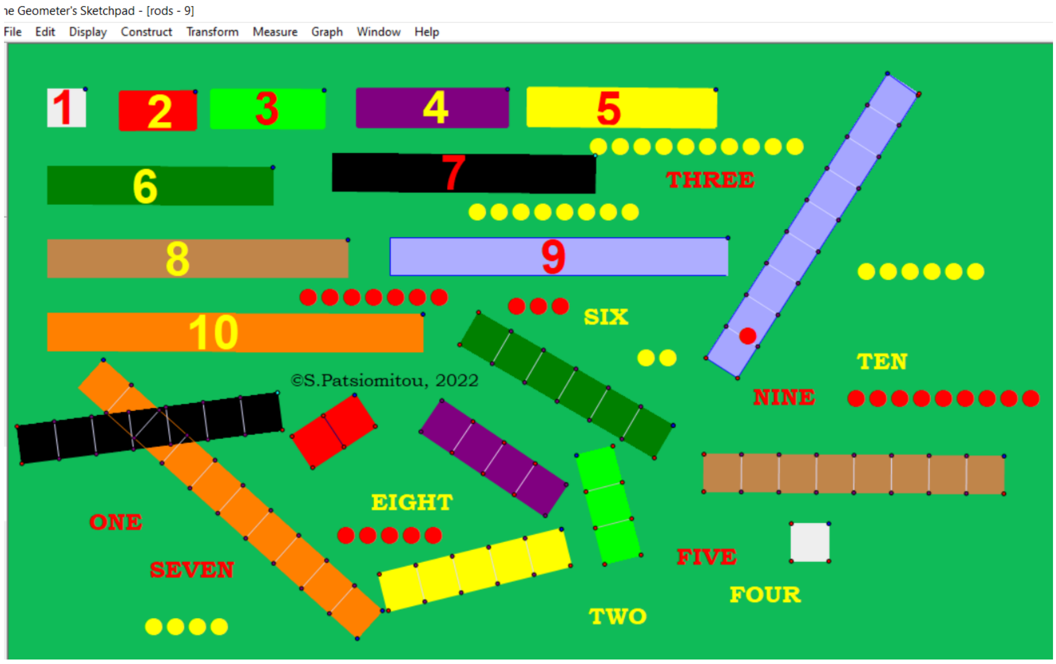

Fig. 17: Multiple representations for the DGS Cui-Rods

Ainsworth supports that there are many difficulties when a student interacts with MERs that has to do with the modality of the representations (e.g., propositional v graphical), the levels of abstraction (e.g., concrete to symbolic representations), the type of representation (e.g., equation, table, line-graph), whether representations are static or dynamic, or differences in labeling and symbols on the representations (Ainsworth, 1999b, p. 34). Ainsworth (2006) argues that

> *"Multiple external representations can provide unique benefits when people are learning complex ideas […] the effectiveness of multiple representations can best be understood by considering three fundamental aspects of learning: the design parameters […], the functions that multiple representations serve in supporting learning and the cognitive tasks that must be undertaken by a learner interacting with multiple representations"* (p. 183)

On the other hand, Yanik, Helding, & Flores (2008) mention students' difficulties in applying numbers as measures on number lines. As they report "the unit and unitization concepts do not develop naturally" (p.693). Moreover, they report "students' confusion about the location of zero." (p.701). As I argued elsewhere (Patsiomitou, 2019b, p.37) a segment (or a line) is a geometrical object. We can create segments in a DGS environment, then measure their length and calculate their sum. We can also use the symbol "+" to represent the process of segments' addition, leading to the concept of segments' sum in geometry. "The case of the addition of two segments in geometry represented by two separate objects identified by two symbols, is more complex, because it includes both a figural and an algebraic entity. The figure of the segment which represents a concrete real "thing" is the figural part; the number which is the measure of the segments' length (or the distance of the endpoints of the segment) represents the algebraic part. In addition, the pupils have to represent the addition of segments with a concrete segment and then represent this action by means of a symbolic representation"

(Patsiomitou, 2019b, p.38). A crucial point is how to give pupils a manipulative that will help them understand the number "zero" as well as negative numbers. Krajcsi et al. (2021, p. 2) report that there is a lack of developmental models for the understanding of "zero". They also argue that "preschoolers are unsure whether zero is a number" (p.15). As they write (Krajcsi et al., 2021): "While knowledge on the development of understanding positive integers is rapidly growing, the development of understanding zero remains not well-understood". (p. 1)

In my opinion, the number "zero" can only be understood in the first class of high school, when the students learn about positive and negative numbers. As we know, in mathematics, zero is classified as an even number. Do students really understand the number zero as an odd or as an even number? Let me provide an example of a discussion included in a study by Schoenfeld & Kilpatrick (2008, p. 21). They report a discussion with third-grade students, among them, Sean. During the lesson Sean said, "[…], if [zero] was an even number, how— what two things could make it?". There is a substantial difference with regard to a pupil's understanding of the number zero in the next example. Papert (1984) in his study "Microworlds: Transforming Education" describes the experience of a little girl who discovered number "zero" as she played with a microworld. This was a crucial point for her understanding, as she understood that the command "S0" made the microworld stop moving. As Papert argues (1984, p. 81):

> *"I think she was excited because she had discovered zero. They tell us in school that the Greek mathematicians, Pythagoras and Euclid and others, these incredibly inventive people, didn't know about zero. […] The fact that not every child discovers zero this way reflects an essential property of the learning process. No two people follow the same path of learnings, discoveries, and revelations. You learn in the deepest way when something happens that makes you fall in love with a particular piece of knowledge."*

5. Transforming Mathematics Education

The mindful use of technology by primary school teachers can attract /engage children in key skills such as play math-games, self-expression, and computational thinking (See also website [18]). New cognitive tools are not included [or included in a very slow way] for the teaching of mathematical concepts. It is particularly important for the 'movement' of a process by applying innovative practices to change the negative views that a large portion of teachers have regarding technology. This seems to focus on a lack of knowledge because of the phobias surrounding technological tools in the mathematics classroom, leading to an adherence to traditional teaching methods. Clements & Mcmillen (1996) argue that *"computer manipulatives allow for changing the arrangement or representation, link the concrete and the symbolic by means of feedback and dynamically link multiple representations"* (p.272-274).

Students' understanding of meanings often led me to note the sequence of steps or stages through which they gathered information from the [computing] environment as stimuli. The information from the computer environment goes through a modification, linked to students' minds stored information (or is modified in the light of the information stored in their mind) so they can answer the teacher's questions or participate in a class discussion.

In general, the whole issue has to do with the way the teachers/researchers /or educators perceive the world, the natural objects (unconsciously), how they compare them mentally (consciously) with theoretical constructs of mathematics in order to represent them and how they instrumentally decode (Patsiomitou, 2011) them using technology.

According to Fangchun Zhu (2020) *"DGS not only affects the learning process of the students but also affects the teaching methods at the same time"* (p. 21). In my opinion, a dynamic geometry environment not only impacts on the learning process, it also affects the construction of *instrumental learning trajectories* (Patsiomitou, 2021) and, consequently, the students' knowledge. In other words, a teacher's knowledge and *instrumental decoding competence* affects their pupils' learning of mathematics. Furthermore, the researcher-teacher's instrumental decoding competence leads to transformations in elements of the instructional path, adding dynamic transformations that can help learners transform their knowledge efficiently.

Finally, it is important to continue teaching and research concepts in this vital field, through activities and tasks that involve children in the learning process, so using linking visual representations (Patsiomitou, 2008, 2019a, b) they will learn how to develop, interpret, and make sense of math-concepts. This argument recognizes and underlines the force of Kant's argument (1929/1965), that: *There can be no doubt that all our knowledge begins with experience.* […] "Understanding is the faculty of knowledge and […] knowledge consists in the determinate relation of given representations to an object". What is important to investigate is the level of strength of these links or connections in the students' mind, which can illustrate how the learning of concepts was accomplished. Future investigation is necessary to explore this issue further.

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